

Gauge Field Theory Coherent States (GCS) : III. Ehrenfest Theorems

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Abstract

In the preceding paper of this series of articles we established peakedness properties of a family of coherent states that were introduced by Hall for any compact gauge group and were later generalized to gauge field theory by Ashtekar, Lewandowski, Marolf, Mourão and Thiemann.

In this paper we establish the “Ehrenfest Property” of these states which are labelled by a point (A, E) , a connection and an electric field, in the classical phase space. By this we mean that

- i) The expectation value of *all* elementary quantum operators \hat{O} with respect to the coherent state with label (A, E) is given to zeroth order in \hbar by the value of the corresponding classical function O evaluated at the phase space point (A, E) and
- ii) The expectation value of the commutator between two elementary quantum operators $[\hat{O}_1, \hat{O}_2]/(i\hbar)$ divided by $i\hbar$ with respect to the coherent state with label (A, E) is given to zeroth order in \hbar by the value of the Poisson bracket between the corresponding classical functions $\{O_1, O_2\}$ evaluated at the phase space point (A, E) .

These results can be extended to all polynomials of elementary operators and to a certain non-polynomial function of the elementary operators associated with the volume operator of quantum general relativity. It follows that the infinitesimal quantum dynamics of quantum general relativity is to zeroth order in \hbar indeed given by classical general relativity.

1 Introduction

Quantum General Relativity (QGR) has matured over the past decade to a mathematically well-defined theory of quantum gravity. In contrast to string theory, by definition QGR is a manifestly background independent, diffeomorphism invariant and non-perturbative theory. The obvious advantage is that one will never have to postulate the existence of a non-perturbative extension of the theory, which in string theory has been called the still unknown M(ystery)-Theory.

The disadvantage of a non-perturbative and background independent formulation is, of course, that one is faced with new and interesting mathematical problems so that one cannot just go ahead and “start calculating scattering amplitudes”: As there is no

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background around which one could perturb, rather the full metric is fluctuating, one is not doing quantum field theory on a spacetime but only on a differential manifold. Once there is no (Minkowski) metric at our disposal, one loses familiar notions such as causality, locality, Poincaré group and so forth, in other words, the theory is not a theory to which the Wightman axioms apply. Therefore, one must build an entirely new mathematical apparatus to treat the resulting quantum field theory which is *drastically different from the Fock space picture to which particle physicists are used to*.

As a consequence, the mathematical formulation of the theory was the main focus of research in the field over the past decade. The main achievements to date are the following (more or less in chronological order) :

i) *Kinematical Framework*

The starting point was the introduction of new field variables [1] for the gravitational field which are better suited to a background independent formulation of the quantum theory than the ones employed until that time. In its original version these variables were complex valued, however, currently their real valued version, considered first in [2] for *classical* Euclidean gravity and later in [3] for *classical* Lorentzian gravity, is preferred because to date it seems that it is only with these variables that one can rigorously define the kinematics and dynamics of Euclidean or Lorentzian *quantum* gravity [4].

These variables are coordinates for the infinite dimensional phase space of an $SU(2)$ gauge theory subject to further constraints besides the Gauss law, that is, a connection and a canonically conjugate electric field. As such, it is very natural to introduce smeared functions of these variables, specifically Wilson loop and electric flux functions. (Notice that one does not need a metric to define these functions, that is, they are background independent). This had been done for ordinary gauge fields already before in [5] and was then reconsidered for gravity (see e.g. [6]).

The next step was the choice of a representation of the canonical commutation relations between the electric and magnetic degrees of freedom. This involves the choice of a suitable space of distributional connections [7] and a faithful measure thereon [8] which, as one can show [9], is σ -additive. The proof that the resulting Hilbert space indeed solves the adjointness relations induced by the reality structure of the classical theory as well as the canonical commutation relations induced by the symplectic structure of the classical theory can be found in [10]. Independently, a second representation, called the loop representation, of the canonical commutation relations had been advocated (see e.g. [11] and especially [12] and references therein) but both representations were shown to be unitarily equivalent in [13] (see also [14] for a different method of proof).

This is then the first major achievement : The theory is based on a rigorously defined kinematical framework.

ii) *Geometrical Operators*

The second major achievement concerns the spectra of positive semi-definite, self-adjoint geometrical operators measuring lengths [15], areas [16, 17] and volumes [16, 18, 19, 20, 11] of curves, surfaces and regions in spacetime. These spectra are pure point (discrete) and imply a discrete Planck scale structure. It should be pointed out that the discreteness is, in contrast to other approaches to quantum gravity, not put in by hand but it is a *prediction* !

iii) *Regularization- and Renormalization Techniques*

The third major achievement is that there is a new regularization and renormalization technique [21, 22] for diffeomorphism covariant, density-one-valued operators at

our disposal which was successfully tested in model theories [23]. This technique can be applied, in particular, to the standard model coupled to gravity [24, 25] and to the Poincaré generators at spatial infinity [26]. In particular, it works for *Lorentzian* gravity while all earlier proposals could at best work in the Euclidean context only (see, e.g. [12] and references therein). The algebra of important operators of the resulting quantum field theories was shown to be consistent [27]. Most surprisingly, these operators are *UV and IR finite* ! Notice that this result, at least as far as these operators are concerned, is stronger than the believed but unproved finiteness of scattering amplitudes order by order in perturbation theory of the five critical string theories, in a sense we claim that the perturbation series converges. The absence of the divergences that usually plague interacting quantum fields propagating on a Minkowski background can be understood intuitively from the diffeomorphism invariance of the theory : “short and long distances are gauge equivalent”. We will elaborate more on this point in future publications.

iv) *Spin Foam Models*

After the construction of the densely defined Hamiltonian constraint operator of [21, 22], a formal, Euclidean functional integral was constructed in [28] and gave rise to the so-called spin foam models (a spin foam is a history of a graph with faces as the history of edges) [29]. Spin foam models are in close connection with causal spin-network evolutions [30], state sum models [31] and topological quantum field theory, in particular BF theory [32]. To date most results are at a formal level and for the Euclidean version of the theory only but the programme is exciting since it may restore manifest four-dimensional diffeomorphism invariance which in the Hamiltonian formulation is somewhat hidden.

- v) Finally, the fifth major achievement is the existence of a rigorous and satisfactory framework [33, 34, 35, 36, 37, 38, 39] for the quantum statistical description of black holes which reproduces the Bekenstein-Hawking Entropy-Area relation and applies, in particular, to physical Schwarzschild black holes while stringy black holes so far are under control only for extremal charged black holes.

Summarizing, the work of the past decade has now culminated in a promising starting point for a quantum theory of the gravitational field plus matter and the stage is set to pose and answer physical questions.

The most basic and most important question that one should ask is : *Does the theory have classical general relativity as its classical limit ?* Notice that even if the answer is negative, the existence of a consistent, interacting, diffeomorphism invariant quantum field theory in four dimensions is already a quite non-trivial result. However, we can claim to have a satisfactory quantum theory of Einstein’s theory only if the answer is positive.

To settle this issue we have launched an attack based on coherent states which has culminated in a series of papers called “Gauge Field Theory Coherent States” [40, 41, 42, 43, 44, 45] and this paper is the third one of this collection (to be continued). It is closely connected with the companion paper [41]. In [41] we established peakedness properties of the coherent states of the heat kernel family introduced by Hall [46] for arbitrary compact gauge groups which were later applied to gauge field theories in [47]. The results of [41] rest on the explicit determination of the configuration space complexification via the complexifier framework [48]. They reveal that the heat kernel family more or less has all the properties that one would like coherent states to have and that one is used to from the harmonic oscillator coherent states. In particular, these states ψ_m^t are labelled by a point $m = (q, p)$ in the classical phase space and 1) are eigenstates of certain annihilation operators, 2) are overcomplete, 3) saturate the unquenched Heisenberg uncertainty bound

and 4) are peaked in the configuration representation at $x = q$, in the momentum representation at $k = p$ and in the Bargmann-Segal representation at $z = q - ip \simeq m$. Here t is a classicality parameter proportional to Planck's constant \hbar and the peak is the sharper (the decay width $\propto \sqrt{t}$ the smaller) the smaller t and resembles almost a Gaussian.

The properties listed ensure that normal ordered products of creation and annihilation operators have exactly the expectation value, with respect to ψ_m^t , given by the product of the associated classical functions evaluated at the phase space point m *without any quantum corrections*. However, to establish that this expectation value property also holds with respect to the elementary operators in terms of which important operators of quantum gauge field theory, such as Hamiltonians, are formulated is not granted a priori. The problem arises because the creation and annihilation operators of [41] are not polynomial functions of the elementary operators which in turn is directly related to the kinematically non-linear nature of the theory.. Therefore, the framework of [49] to prove Ehrenfest theorems and the determination of the classical limit by using the harmonic oscillator coherent states does not extend to our case since the methods of [49] crucially rest on the assumption that the basic operators are *linear combinations* of creation and annihilation operators.

The present paper is devoted to filling this gap. As in [41] all the proofs will be carried out for the case of rank one gauge groups, that is $G = SU(2), U(1)$, only but by the arguments given in [41] they should readily extend to the case of an arbitrary compact gauge group which we leave for future work [50]. With this restriction, the main result of the present article is that the Ehrenfest property, to zeroth and first order, indeed holds for our coherent states. In other words, the expectation values of polynomials of the elementary operators as well as of an important operator, associated with the volume operator of quantum general relativity mentioned above, which is not a polynomial (not even analytical !) function of the elementary operators, reproduce, to zeroth order in t , the values of the corresponding classical functions at the phase space point given by the coherent state. Moreover, the expectation values of commutators divided by it reproduces the corresponding Poisson bracket, to zeroth order in t , at the given phase space point. These results imply that the quantum dynamics of the operators constructed in [21, 22, 24], as expected, reproduce the infinitesimal classical dynamics of general relativity [51], putting the worries raised in [52] ad acta.

The architecture of the present article is as follows :

Section two summarizes the classical and quantum kinematical framework for diffeomorphism invariant quantum gauge field theories.

In section three, after a brief review of the heat kernel coherent states, we prove the above mentioned Ehrenfest theorems for the case of the gauge-variant coherent states for the gauge group $G = SU(2)$. As stated already in [41], we are mostly interested in gauge-variant coherent states because a) only those manage to verify the satisfaction of a consistent quantum constraint algebra and b) the expectation values of gauge – and diffeomorphism invariant operators are gauge – and diffeomorphism invariant since both gauge groups are represented unitarily on the Hilbert space.

Finally, in appendix A we repeat the analysis of section three for the gauge group $G = U(1)$. As in [41], the Abelian nature of $U(1)$ shortens all the proofs given for $SU(2)$ by an order of magnitude and the reader is urged to study first the appendix before delving into the technically much harder section three.

2 Kinematical Structure of Diffeomorphism Invariant Quantum Gauge Theories

In this section we will recall the main ingredients of the mathematical formulation of (Lorentzian) diffeomorphism invariant classical and quantum field theories of connections with local degrees of freedom in any dimension and for any compact gauge group. See [10, 53] and references therein for more details. In this section we will take all quantities to be dimensionless. More about dimensionful constants will be said in section 3.

2.1 Classical Theory

Let G be a compact gauge group, Σ a D -dimensional manifold admitting a principal G -bundle with connection over Σ . Let us denote the pull-back to Σ of the connection by local sections by A_a^i where $a, b, c, \dots = 1, \dots, D$ denote tensorial indices and $i, j, k, \dots = 1, \dots, \dim(G)$ denote indices for the Lie algebra of G . Likewise, consider a density-one vector bundle of electric fields, whose pull-back to Σ by local sections (their Hodge dual is a $D - 1$ form) is a Lie algebra valued vector density of weight one. We will denote the set of generators of the rank $N - 1$ Lie algebra of G by τ_i which are normalized according to $\text{tr}(\tau_i \tau_j) = -N \delta_{ij}$ and $[\tau_i, \tau_j] = 2 f_{ij}^k \tau_k$ defines the structure constants of $\text{Lie}(G)$.

Let F_i^a be a Lie algebra valued vector density test field of weight one and let f_a^i be a Lie algebra valued covector test field. We consider the smeared quantities

$$F(A) := \int_{\Sigma} d^D x F_i^a A_a^i \text{ and } E(f) := \int_{\Sigma} d^D x E_i^a f_a^i \quad (2.1)$$

While both of them are diffeomorphism covariant, only the latter is gauge covariant, one reason to introduce the singular smearing functions discussed below. The choice of the space of pairs of test fields $(F, f) \in \mathcal{S}$ depends on the boundary conditions on the space of connections and electric fields which in turn depends on the topology of Σ and will not be specified in what follows.

The set of all pairs of smooth functions (A, E) on Σ such that (2.1) is well defined for any $(F, f) \in \mathcal{S}$ defines an infinite dimensional set M . We define a topology on M through the following globally defined metric :

$$d_{\rho, \sigma}[(A, E), (A', E')] \quad (2.2)$$

$$:= \sqrt{-\frac{1}{N} \int_{\Sigma} d^D x [\sqrt{\det(\rho)} \rho^{ab} \text{tr}([A_a - A'_a][A_b - A'_b]) + \frac{[\sigma_{ab} \text{tr}([E^a - E'^a][E^b - E'^b])]}{\sqrt{\det(\sigma)}}]}$$

where ρ_{ab}, σ_{ab} are fiducial metrics on Σ of everywhere Euclidean signature. Their fall-off behaviour has to be suited to the boundary conditions of the fields A, E at spatial infinity. Notice that the metric (2.2) on M is gauge invariant. It can be used in the usual way to equip M with the structure of a smooth, infinite dimensional differential manifold modelled on a Banach (in fact Hilbert) space \mathcal{E} where $\mathcal{S} \times \mathcal{S} \subset \mathcal{E}$. (It is the weighted Sobolev space $H_{0, \rho}^2 \times H_{0, \sigma-1}^2$ in the notation of [54]).

Finally, we equip M with the structure of an infinite dimensional symplectic manifold through the following strong (in the sense of [55]) symplectic structure

$$\Omega((f, F), (f', F'))_m := \int_{\Sigma} d^D x [F_i^a f_a^{i'} - F_i^{a'} f_a^i](x) \quad (2.3)$$

for any $(f, F), (f', F') \in \mathcal{E}$. We have abused the notation by identifying the tangent space to M at m with \mathcal{E} . To prove that Ω is a strong symplectic structure one uses standard

Banach space techniques. Computing the Hamiltonian vector fields (with respect to Ω) of the functions $E(f), F(A)$ we obtain the following elementary Poisson brackets

$$\{E(f), E(f')\} = \{F(A), F'(A)\} = 0, \{E(f), A(F)\} = F(f) \quad (2.4)$$

As a first step towards quantization of the symplectic manifold (M, Ω) one must choose a polarization. As usual in gauge theories, we will use a particular real polarization, specifically connections as the configuration variables and electric fields as canonically conjugate momenta. As a second step one must decide on a complete set of coordinates of M which are to become the elementary quantum operators. The analysis just outlined suggests to use the coordinates $E(f), F(A)$. However, the well-known immediate problem is that these coordinates are not gauge covariant. Thus, we proceed as follows :

Let Γ_0^ω be the set of all piecewise analytic, finite, oriented graphs γ embedded into Σ and denote by $E(\gamma)$ and $V(\gamma)$ respectively its sets of oriented edges e and vertices v respectively. Here finite means that $E(\gamma)$ is a finite set. (One can extend the framework to Γ_0^ω , the restriction to webs of the set of piecewise smooth graphs [56, 57] but the description becomes more complicated and we refrain from doing this here). It is possible to consider the set Γ_σ^ω of piecewise analytic, infinite graphs with an additional regularity property [44] but for the purpose of this paper it will be sufficient to stick to Γ_0^ω . The subscript $_0$ as usual denotes “of compact support” while $_\sigma$ denotes “ σ -finite”.

We denote by $h_e(A)$ the holonomy of A along e and say that a function f on \mathcal{A} is cylindrical with respect to γ if there exists a function f_γ on $G^{|E(\gamma)|}$ such that $f = p_\gamma^* f_\gamma = f \circ p_\gamma$ where $p_\gamma(A) = \{h_e(A)\}_{e \in E(\gamma)}$. Holonomies are invariant under reparameterizations of the edge and in this article we assume that the edges are always analyticity preserving diffeomorphic images from $[0, 1]$ to a one-dimensional submanifold of Σ . Gauge transformations are functions $g : \Sigma \mapsto G$; $x \mapsto g(x)$ and they act on holonomies as $h_e \mapsto g(e(0))h_e g(e(1))$.

Next, given a graph γ we choose a polyhedral decomposition P_γ of Σ dual to γ . The precise definition of a dual polyhedral decomposition can be found in [53] but for the purposes of the present paper it is sufficient to know that P_γ assigns to each edge e of γ an open “face” S_e (a polyhedron of codimension one embedded into Σ) with the following properties :

- (1) the surfaces S_e are mutually non-intersecting,
- (2) only the edge e intersects S_e , the intersection is transversal and consists only of one point which is an interior point of both e and S_e ,
- (3) S_e carries the orientation which agrees with the orientation of e .

Furthermore, we choose a system Π_γ of paths $\rho_e(x) \subset S_e$, $x \in S_e$, $e \in E(\gamma)$ connecting the intersection point $p_e = e \cap S_e$ with x . The paths vary smoothly with x and the triples $(\gamma, P_\gamma, \Pi_\gamma)$ have the property that if γ, γ' are diffeomorphic, so are $P_\gamma, P_{\gamma'}$ and $\Pi_\gamma, \Pi_{\gamma'}$, see [53] for details.

With these structures we define the following function on (M, Ω)

$$P_i^e(A, E) := -\frac{1}{N} \text{tr}(\tau_i h_e(0, 1/2) [\int_{S_e} h_{\rho_e(x)} * E(x) h_{\rho_e(x)}^{-1}] h_e(0, 1/2)^{-1}) \quad (2.5)$$

where $h_e(s, t)$ denotes the holonomy of A along e between the parameter values $s < t$, $*$ denotes the Hodge dual, that is, of $*E$ is a $(D-1)$ -form on Σ , $E^a := E_i^a \tau_i$ and we have chosen a parameterization of e such that $p_e = e(1/2)$.

Notice that in contrast to similar variables used earlier in the literature the function P_i^e is *gauge covariant*. Namely, under gauge transformations it transforms as $P^e \mapsto g(e(0))P^e g(e(0))^{-1}$, the price to pay being that P^e depends on both A and E and not only on E . The idea is therefore to use the variables h_e, P_i^e for all possible graphs γ as the coordinates of M .

The problem with the functions $h_e(A)$ and $P_i^e(A, E)$ on M is that they are not differentiable on M , that is, Dh_e, DP_i^e are nowhere bounded operators on \mathcal{E} as one can easily see. The reason for this is, of course, that these are functions on M which are not properly smeared with functions from \mathcal{S} , rather they are smeared with distributional test functions with support on e or S_e respectively. Nevertheless one would like to base the quantization of the theory on these functions as basic variables because of their gauge and diffeomorphism covariance. Indeed, under diffeomorphisms $h_e \mapsto h_{\varphi^{-1}(e)}, P_i^e \mapsto P_i^{\varphi^{-1}(e)}$ where the latter notation means that $P_i^{\varphi^{-1}(e)}$ is labelled by $\varphi^{-1}(S_e), \varphi^{-1}(\Pi_\gamma)$. We proceed as follows.

Definition 2.1 By \bar{M}_γ we denote the direct product $[G \times \text{Lie}(G)]^{|E(\gamma)|}$. The subset of \bar{M}_γ of pairs $(h_e(A), P_i^e(A, E))_{e \in E(\gamma)}$ as (A, E) varies over M will be denoted by $(\bar{M}_\gamma)|_M$. We have a corresponding pull-back map $p_\gamma : M \mapsto \bar{M}_\gamma$ which maps M onto $(\bar{M}_\gamma)|_M$.

Notice that the set $(\bar{M}_\gamma)|_M$ is in general a proper subset of \bar{M}_γ , depending on the boundary conditions on (A, E) , the topology of Σ and the “size” of e, S_e . For instance, in the limit of $e, S_e \rightarrow e \cap S_e$ but holding the number of edges fixed, $(\bar{M}_\gamma)|_M$ will consist of only one point in M_γ . This follows from the smoothness of the (A, E) .

We equip a subset M_γ of \bar{M}_γ with the structure of a differentiable manifold modelled on the Banach space $\mathcal{E}_\gamma = \mathbb{R}^{2 \dim(G)|E(\gamma)|}$ by using the natural direct product manifold structure of $[G \times \text{Lie}(G)]^{|E(\gamma)|}$. While \bar{M}_γ is a kind of distributional phase space, M_γ satisfies suitable regularity properties similar to M .

In order to proceed and to give M_γ a symplectic structure *derived from* (M, Ω) one must regularize the elementary functions h_e, P_i^e by writing them as limits (in which the regulator vanishes) of functions which can be expressed in terms of the $F(A), E(f)$. Then one can compute their Poisson brackets with respect to the symplectic structure Ω at finite regulator and then take the limit pointwise on M . The result is the following well-defined strong symplectic structure Ω_γ on M_γ .

$$\begin{aligned} \{h_e, h_{e'}\}_\gamma &= 0 \\ \{P_i^e, h_{e'}\}_\gamma &= \delta_{e'}^e \frac{\tau_i}{2} h_e \\ \{P_i^e, P_j^{e'}\}_\gamma &= -\delta^{ee'} f_{ij}{}^k P_k^e \end{aligned} \tag{2.6}$$

Since Ω_γ is obviously block diagonal, each block standing for one copy of $G \times \text{Lie}(G)$, to check that Ω_γ is non-degenerate and closed reduces to doing it for each factor together with an appeal to well-known Hilbert space techniques to establish that Ω_γ is a surjection of \mathcal{E}_γ . This is done in [53] where it is shown that each copy is isomorphic with the cotangent bundle T^*G equipped with the symplectic structure (2.6) (choose $e = e'$ and delete the label e).

Now that we have managed to assign to each graph γ a symplectic manifold $(M_\gamma, \Omega_\gamma)$ we can quantize it by using geometric quantization. This can be done in a well-defined way because the relations (2.6) show that the corresponding operators are non-distributional. This is therefore a clean starting point for the regularization of any operator of quantum gauge field theory which can always be written in terms of the \hat{h}_e, \hat{P}^e , $e \in E(\gamma)$ if we apply this operator to a function which depends only on the h_e , $e \in E(\gamma)$.

The question is what $(M_\gamma, \Omega_\gamma)$ has to do with (M, Ω) . In [53] it is shown that there exists a partial order \prec on the set \mathcal{L} of triples $l = (\gamma, P_\gamma, \Pi_\gamma)$. In particular, $\gamma \prec \gamma'$ means $\gamma \subset \gamma'$ and \mathcal{L} is a directed set so that one can form a generalized projective limit M_∞ of the M_γ (we abuse notation in displaying the dependence of M_γ on γ only rather than on l). For this one verifies that the family of symplectic structures Ω_γ is self-consistent

in the sense that if $(\gamma, P_\gamma, \Pi_\gamma) \prec (\gamma', P_{\gamma'}, \Pi_{\gamma'})$ then $p_{\gamma'\gamma}^*\{f, g\}_\gamma = \{p_{\gamma'\gamma}^*f, p_{\gamma'\gamma}^*g\}_{\gamma'}$ for any $f, g \in C^\infty(M_\gamma)$ and $p_{\gamma'\gamma} : M_{\gamma'} \mapsto M_\gamma$ is a system of natural projections, more precisely, of (non-invertible) symplectomorphisms.

Now, via the maps p_γ of definition 2.1 we can identify M with a subset of M_∞ . Moreover, in [53] it is shown that there is a generalized projective sequence $(\gamma_n, P_{\gamma_n}, \Pi_{\gamma_n})$ such that $\lim_{n \rightarrow \infty} p_{\gamma_n}^* \Omega_{\gamma_n} = \Omega$ pointwise in M . This displays (M, Ω) as embedded into a projective generalized limit of the $(M_\gamma, \Omega_\gamma)$, intuitively speaking, as γ fills all of Σ , we recover (M, Ω) from the $(M_\gamma, \Omega_\gamma)$. Of course, this works with Γ_0^ω only if Σ is compact, otherwise we need the extension to Γ_σ^ω .

It follows that quantization of (M, Ω) , and conversely taking the classical limit, can be studied purely in terms of $(M_\gamma, \Omega_\gamma)$ for *all* γ . The quantum kinematical framework is given in the next subsection.

2.2 Quantum Theory

Let us denote the set of all smooth connections by \mathcal{A} . This is our classical configuration space and we will choose for its coordinates the holonomies $h_e(A)$, $e \in \gamma$, $\gamma \in \Gamma_0^\omega$. \mathcal{A} is naturally equipped with a metric topology induced by (2.2).

Recall the notion of a function cylindrical over a graph from the previous subsection. A particularly useful set of cylindrical functions are the so-called spin-network functions [58, 59, 13]. A spin-network function is labelled by a graph γ , a set of non-trivial irreducible representations $\vec{\pi} = \{\pi_e\}_{e \in E(\gamma)}$ (choose from each equivalence class of equivalent representations once and for all a fixed representant), one for each edge of γ , and a set $\vec{c} = \{c_v\}_{v \in V(\gamma)}$ of contraction matrices, one for each vertex of γ , which contract the indices of the tensor product $\otimes_{e \in E(\gamma)} \pi_e(h_e)$ in such a way that the resulting function is gauge invariant. We denote spin-network functions as T_I where $I = \{\gamma, \vec{\pi}, \vec{c}\}$ is a compound label. One can show that these functions are linearly independent. From now on we denote by $\tilde{\Phi}_\gamma$ finite linear combinations of spin-network functions over γ , by Φ_γ the finite linear combinations of elements from any possible $\tilde{\Phi}_{\gamma'}$, $\gamma' \subset \gamma$ a subgraph of γ and by Φ the finite linear combinations of spin-network functions over an arbitrary finite collection of graphs. Clearly $\tilde{\Phi}_\gamma$ is a subspace of Φ_γ . To express this distinction we will say that functions in $\tilde{\Phi}_\gamma$ are labelled by the “coloured graphs” γ while functions in Φ_γ are labelled simply by graphs γ where we abuse notation by using the same symbol γ .

The set Φ of finite linear combinations of spin-network functions forms an Abelian $*$ algebra of functions on \mathcal{A} . By completing it with respect to the sup-norm topology it becomes an Abelian C^* algebra (here the compactness of G is crucial). The spectrum $\overline{\mathcal{A}}$ of this algebra, that is, the set of all algebraic homomorphisms $\mathcal{B} \mapsto \mathbb{C}$ is called the quantum configuration space. This space is equipped with the Gel'fand topology, that is, the space of continuous functions $C^0(\overline{\mathcal{A}})$ on $\overline{\mathcal{A}}$ is given by the Gel'fand transforms of elements of \mathcal{B} . Recall that the Gel'fand transform is given by $\tilde{f}(\bar{A}) := \bar{A}(f) \forall \bar{A} \in \overline{\mathcal{A}}$. It is a general result that $\overline{\mathcal{A}}$ with this topology is a compact Hausdorff space. Obviously, the elements of \mathcal{A} are contained in $\overline{\mathcal{A}}$ and one can show that \mathcal{A} is even dense [60]. Generic elements of $\overline{\mathcal{A}}$ are, however, distributional.

The idea is now to construct a Hilbert space consisting of square integrable functions on $\overline{\mathcal{A}}$ with respect to some measure μ . Recall that one can define a measure on a locally compact Hausdorff space by prescribing a positive linear functional χ_μ on the space of continuous functions thereon. The particular measure we choose is given by $\chi_{\mu_0}(\tilde{T}_I) = 1$ if $I = \{\{p\}, \vec{0}, \vec{1}\}$ and $\chi_{\mu_0}(\tilde{T}_I) = 0$ otherwise. Here p is any point in Σ , 0 denotes the trivial representation and 1 the trivial contraction matrix. In other words, (Gel'fand transforms of) spin-network functions play the same role for μ_0 as Wick-polynomials do for Gaussian measures and like those they form an orthonormal basis in the Hilbert space

$\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu_0)$ obtained by completing their finite linear span Φ .

An equivalent definition of $\overline{\mathcal{A}}, \mu_0$ is as follows :

$\overline{\mathcal{A}}$ is in one to one correspondence, via the surjective map H defined below, with the set $\overline{\mathcal{A}'} := \text{Hom}(\mathcal{X}, G)$ of homomorphisms from the groupoid \mathcal{X} of composable, holonomically independent, analytical paths into the gauge group. The correspondence is explicitly given by $\overline{\mathcal{A}} \ni \bar{A} \mapsto H_{\bar{A}} \in \text{Hom}(\mathcal{X}, G)$ where $\mathcal{X} \ni e \mapsto H_{\bar{A}}(e) := \bar{A}(h_e) = \tilde{h}_e(\bar{A}) \in G$ and \tilde{h}_e is the Gel'fand transform of the function $\mathcal{A} \ni A \mapsto h_e(A) \in G$. Consider now the restriction of \mathcal{X} to \mathcal{X}_γ , the groupoid of composable edges of the graph γ . One can then show that the projective limit of the corresponding *cylindrical sets* $\overline{\mathcal{A}'}_\gamma := \text{Hom}(\mathcal{X}_\gamma, G)$ coincides with $\overline{\mathcal{A}'}$. Moreover, we have $\{\{H(e)\}_{e \in E(\gamma)}; H \in \overline{\mathcal{A}'}_\gamma\} = \{\{H_{\bar{A}}(e)\}_{e \in E(\gamma)}; \bar{A} \in \overline{\mathcal{A}}\} = G^{|E(\gamma)|}$. Let now $f \in \mathcal{B}$ be a function cylindrical over γ then

$$\chi_{\mu_0}(\tilde{f}) = \int_{\overline{\mathcal{A}}} d\mu_0(\bar{A}) \tilde{f}(\bar{A}) = \int_{G^{|E(\gamma)|}} \otimes_{e \in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e \in E(\gamma)})$$

where μ_H is the Haar measure on G . As usual, \mathcal{A} turns out to be contained in a measurable subset of $\overline{\mathcal{A}}$ which has measure zero with respect to μ_0 .

Let Φ_γ , as before, be the finite linear span of spin-network functions over γ and \mathcal{H}_γ its completion with respect to μ_0 . Clearly, \mathcal{H} itself is the completion of the finite linear span Φ of vectors from the mutually orthogonal $\tilde{\Phi}_\gamma$. Our basic coordinates of M_γ are promoted to operators on \mathcal{H} with dense domain Φ . As h_e is group-valued and P^e is real-valued we must check that the adjointness relations coming from these reality conditions as well as the Poisson brackets (2.6) are implemented on our \mathcal{H} . This turns out to be precisely the case if we choose \hat{h}_e to be a multiplication operator and $\hat{P}_j^e = i\hbar\kappa X_j^e/2$ where $X_j^e = X_j(h_e)$ and $X^j(h)$, $h \in G$ is the vector field on G generating left translations into the j -th coordinate direction of $\text{Lie}(G) \equiv T_h(G)$ (the tangent space of G at h can be identified with the Lie algebra of G) and κ is the coupling constant of the theory. For details see [10, 53].

3 Ehrenfest Theorems

Let us recall the most important facts from [41].

Instead of working with the quantities P_i^e of section 2 we use the dimensionless objects $p_i^e = P_i^e/a^{n_D}$. If P_i^e is already dimensionless then so is a and we choose $n_D = 1$. Otherwise a is an arbitrary but fixed constant of the dimension $[a] = \text{cm}^1$ and the power n_D is so chosen that p_i^e is dimensionless. In both cases, the numerical value of a is macroscopic, say $a = 1$ or $a = 1\text{cm}$ respectively. The power n_D depends on the dimensionality of Σ and the theory, e.g. $n_D = 2$ for general relativity in $D = 3$ spatial dimensions. Also, if κ is the coupling constant of the theory (the coefficient $1/\kappa$ in front of the classical action) then P_i^e in (2.6) has to be replaced by P_i^e/κ . It follows from the canonical commutation relations that if \hat{h}_e as before is a multiplication operator in the connection representation then $\hat{p}_j^e = itX_j^e/2$ where

$$t := \frac{\alpha}{a^{n_D}} \text{ and } \alpha = \kappa\hbar \quad (3.1)$$

define the classicality parameter and the Feinstruktur constant respectively. For instance, in four-dimensional general relativity $\alpha = \ell_p^2$ is the Planck area and for $a = 1\text{cm}$ we have $\sqrt{t}/a = \ell_p/\text{cm} \approx 10^{-32}$. All our estimates are based on the fact that t is a tiny positive number.

Consider first only one edge e of a graph γ , then we define the complexifier

$$\hat{C}_e := \frac{a^{n_D}}{2\kappa} \delta^{ij} \hat{p}_i^e \hat{p}_j^e \quad (3.2)$$

and the coherent state, labelled by the phase space point $g_e = e^{-ip_j^e \tau_j / 2} h_e \in G^\mathbb{C}$ and the classicality parameter t , by

$$\psi_{e,g_e}^t(h_e) := [e^{-\frac{\hat{C}_e}{\hbar}} \delta_{h'}(h_e)]_{h' \rightarrow g_e} = [e^{t\Delta_e/2} \delta_{h'}(h_e)]_{h' \rightarrow g_e} \quad (3.3)$$

where $\delta_{h'}$ denotes the δ distribution on G with support at h' and in (3.3) one is supposed to analytically continue h' . One can give the following explicit sum formula for $\psi_{e,g_e}^t(h_e)$,

$$\psi_{e,g_e}^t(h_e) = \sum_{\pi} d_{\pi} e^{-t\lambda_{\pi}/2} \chi_{\pi}(g_e h_e^{-1}) \quad (3.4)$$

where the sum is over the equivalence classes of irreducible representations π of G , χ_{π} is the character of π and $-\lambda_{\pi}$ is the eigenvalue of $-\Delta_e$ with eigenfunctions $\chi_{\pi}(g_e h_e^{-1})$. The operator $e^{t\Delta_e/2}$ is sometimes called the heat kernel operator. Notice that the states ψ_{e,g_e}^t are not normalized.

The generalization to the whole graph γ is straightforward, it is simply given by the product over edges

$$\psi_{\gamma,g_{\gamma}}^t(h_{\gamma}) = \prod_{e \in E(\gamma)} \psi_{e,g_e}^t(h_e) \quad (3.5)$$

where $g_{\gamma} = \{g_e\}_{e \in E(\gamma)}$, $h_{\gamma} = \{h_e\}_{e \in E(\gamma)}$. The product states (3.5) are obtained by applying the operator $\exp(t\Delta_{\gamma}/2)$, $\Delta_{\gamma} = \sum_{e \in E(\gamma)} \Delta_e$ to the product delta – distribution

$$\delta_{\gamma,h'_{\gamma}}(h_{\gamma}) = \prod_{e \in E(\gamma)} \delta_{h'_e}(h_e) \quad (3.6)$$

followed by analytical continuation. This formula is meaningful only if $\gamma \in \Gamma_0^{\omega}$ is a finite graph, for truly infinite graphs in Γ_{σ}^{ω} we must work immediately with products of normalized coherent states as otherwise such states would not be normalizable. For the purpose of this paper it will be sufficient to stick with Γ_0^{ω} for the following reason. Since the Poisson brackets (2.6) are promoted to the following commutation relations

$$\begin{aligned} [\hat{h}_e, \hat{h}_{e'}] &= 0 \\ [\hat{p}_j^e, \hat{h}_{e'}] &= it\delta_{e'}^e \frac{\tau_j}{2} \hat{h}_e \\ [\hat{p}_i^e, \hat{p}_j^{e'}] &= -it\delta^{ee'} f_{ij}^k \hat{p}_k^e \end{aligned} \quad (3.7)$$

on the Hilbert space \mathcal{H}_{γ} , the completion of Φ_{γ} , it follows that due to the commutativity of operators labelled by different edges every polynomial of the elementary operators $\{\hat{h}_e, \hat{p}_j^e\}_{e \in E(\gamma)}$ can in fact be reduced to sums of monomials of the form

$$\hat{O}_{\gamma} = \prod_{e \in E(\gamma)} \hat{O}_e \quad (3.8)$$

where for each e the operator $\hat{O}_e = \hat{O}_e(\hat{h}_e, \hat{p}_j^e)$ is a certain polynomial of the $2 \dim(G)$ independent operators $(\hat{h}_e)_{AB}, \hat{p}_j^e$ for the same e (A, B, C, \dots are group indices). Obviously the order of the operators \hat{O}_e is irrelevant but not the order of the elementary operators appearing in \hat{O}_e . It follows that the expectation value of (infinite) sums of monomials of the type (3.8) is given by the same sum over expectation values of monomials and the latter have the following simple product structure with respect to the state (3.5)

$$\frac{\langle \psi_{\gamma,g_{\gamma}}^t, \hat{O}_{\gamma} \psi_{\gamma,g_{\gamma}}^t \rangle}{\|\psi_{\gamma,g_{\gamma}}^t\|^2} = \prod_{e \in E(\gamma)} \frac{\langle \psi_{e,g_e}^t, \hat{O}_e \psi_{e,g_e}^t \rangle}{\|\psi_{e,g_e}^t\|^2} \quad (3.9)$$

Also, as far as commutators are concerned, notice the commutator formula for monomial operators $\hat{O}_\gamma, \hat{O}'_\gamma$ of the type (3.8) given by

$$[\hat{O}_\gamma, \hat{O}'_\gamma] = \sum_{e \in E(\gamma)} [\hat{O}_e, \hat{O}'_e] \prod_{e' \in E(\gamma) - \{e\}} (\hat{O}_{e'} \hat{O}'_{e'}) \quad (3.10)$$

which can be proved by complete induction over $|E(\gamma)|$. Thus, commutators of monomials again reduce to sums over monomials. We summarize these simple observations by the following theorem.

Theorem 3.1 *Let $\gamma \in \Gamma_0^\omega$ be a graph, $g_\gamma \in M_\gamma$ a point in the phase space and $\hat{O}_\gamma, \hat{O}'_\gamma$ monomial operators. Suppose that for each $e \in E(\gamma)$ we have*

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\langle \psi_{e,g_e}^t, \hat{O}_e \psi_{e,g_e}^t \rangle}{\|\psi_{e,g_e}^t\|^2} &= O_e(h_e(g_e), p_j^e(g_e)) \\ \lim_{t \rightarrow 0} \frac{\langle \psi_{e,g_e}^t, \frac{[\hat{O}_e, \hat{O}'_e]}{it} \psi_{e,g_e}^t \rangle}{\|\psi_{e,g_e}^t\|^2} &= \{O_e, O'_e\}_e(h_e(g_e), p_j^e(g_e)) \end{aligned} \quad (3.11)$$

where the polar decomposition $g_e = H_e(g_e)h_e(g_e)$, $H_e(g_e) = \exp(-i\tau_j p_j^e(g_e)/2)$ specifies $h_e(g_e), p_j^e(g_e)$ uniquely. Then

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\langle \psi_{\gamma,g_\gamma}^t, \hat{O}_\gamma \psi_{\gamma,g_\gamma}^t \rangle}{\|\psi_{\gamma,g_\gamma}^t\|^2} &= O_\gamma(h_\gamma(g_\gamma), p_j^\gamma(g_\gamma)) \\ \lim_{t \rightarrow 0} \frac{\langle \psi_{\gamma,g_\gamma}^t, \frac{[\hat{O}_\gamma, \hat{O}'_\gamma]}{it} \psi_{\gamma,g_\gamma}^t \rangle}{\|\psi_{\gamma,g_\gamma}^t\|^2} &= \{O_\gamma, O'_\gamma\}_\gamma(h_\gamma(g_\gamma), p_j^\gamma(g_\gamma)) \end{aligned} \quad (3.12)$$

where $p_j^\gamma = \{p_j^e\}_{e \in E(\gamma)}$.

This theorem shows that in order to establish Ehrenfest theorems it will be completely sufficient to consider the problem *for one copy of the group only*. This is even true in the extension from Γ_0^ω to Γ_σ^ω because the operators that appear in applications can be written as infinite sums of monomials *each of which depends on a finite subgraph of an infinite graph only*. However, if $\Gamma_0^\omega \ni \gamma \subset \gamma' \in \Gamma_\sigma^\omega$ then we can write a given monomial operator \hat{O}_γ also as $\hat{O}_{\gamma'} = \hat{O}_\gamma \cdot \prod_{e' \in E(\gamma') - \gamma} 1_{e'}$ where $1_{e'}$ denotes the unit operator on $\mathcal{H}_{e'}$. Thus, we get for the expectation values

$$\frac{\langle \psi_{\gamma',g_{\gamma'}}^t, \hat{O}_{\gamma'} \psi_{\gamma',g_{\gamma'}}^t \rangle}{\|\psi_{\gamma',g_{\gamma'}}^t\|^2} = \frac{\langle \psi_{\gamma,g_\gamma}^t, \hat{O}_\gamma \psi_{\gamma,g_\gamma}^t \rangle}{\|\psi_{\gamma,g_\gamma}^t\|^2} \quad (3.13)$$

so the problem reduces again to one for $\gamma \in \Gamma_0^\omega$.

Equation (3.13) seems to indicate that an extension of coherent states to infinite graphs is not really necessary. However, if Σ is non-compact then the only way to approximate, say a classical metric in general relativity which is everywhere non-degenerate, by coherent states over graphs $\gamma \in \Gamma_0^\omega$ is by using a countably infinite superposition of them, say $\hat{\psi}_{\gamma,g_\gamma}^t = \sum_n z_n \psi_{\gamma_n,g_{\gamma_n}}^t$, $z_n \in \mathbb{C}, n \in \mathbb{N}$ where $\gamma = \cup_n \gamma_n$ and the $\psi_{\gamma_n,g_{\gamma_n}}^t$ are the coherent states (3.5). Suppose now that we consider the following operator $\hat{O}_\gamma = \sum_n \hat{O}_{\gamma_n}$ over γ . In applications it is usually true that the γ_n are mutually disjoint but they fill Σ everywhere with respect to the metric to be approximated, that is, γ fills Σ . Now the $\psi_{\gamma_n,g_{\gamma_n}}^t$ are not mutually orthogonal, rather for $n \neq m$ we have $\langle \psi_{\gamma_m,g_{\gamma_m}}^t, \psi_{\gamma_n,g_{\gamma_n}}^t \rangle = 1$ while $\|\psi_{\gamma_m,g_{\gamma_m}}^t\| > 1$ for all g_γ, t . Now in applications it turns out that

$$\langle \psi_{\gamma_m,g_{\gamma_m}}^t, \hat{O}_{\gamma_n} \psi_{\gamma_p,g_{\gamma_p}}^t \rangle = \delta_{m,n} \delta_{n,p} \langle \psi_{\gamma_n,g_{\gamma_n}}^t, \hat{O}_{\gamma_n} \psi_{\gamma_n,g_{\gamma_n}}^t \rangle$$

and thus indeed

$$\langle \tilde{\psi}_{\gamma, g_\gamma}^t, \hat{O}_\gamma \tilde{\psi}_{\gamma, g_\gamma}^t \rangle = \sum_n |z_n|^2 \langle \psi_{\gamma_n, g_{\gamma_n}}^t, \hat{O}_{\gamma_n} \psi_{\gamma_n, g_{\gamma_n}}^t \rangle$$

yields the correct expectation value provided that $|z_n| = 1/||\psi_{\gamma_n, g_{\gamma_n}}^t||$. However, then the norm squared of $\tilde{\psi}_{\gamma, g_\gamma}^t$ is given by

$$||\tilde{\psi}_{\gamma, g_\gamma}^t||^2 = [\sum_n 1] + 2 \sum_{m < n} \Re(\bar{z}_m z_n)$$

which is divergent. Thus, the only way to deal with semiclassical physics in the case that Σ is non-compact is to use the extension to infinite graphs Γ_σ^ω .

The remainder of this section then is subdivided into two parts. In the first one we prove the Ehrenfest Theorem for the polynomial algebra of operators for one copy of the group only for the case of $G = SU(2)$. In the second we use these results to extend the theorem to a certain class of operators which are not polynomial functions of the elementary operators and mix operators labelled by different edges by making an appeal to the moment problem by Hamburger for measures.

3.1 Polynomial Algebra of Operators over One Edge

As the problem is isomorphic for all the edges of a graph, we drop the label e in this subsection and deal only with the operators \hat{h}_{AB}, \hat{p}_j ; $A, B, C, \dots \in \{-1/2, 1/2\}$; $j, k, l, \dots = 1, 2, 3$ which obey the CCR algebra (3.7) with the label $e = e'$ dropped.

This subsection is divided into four parts. In the first we reduce the computation of expectation values of operator monomials to the computation of matrix elements of elementary operators between coherent states. In the second we estimate the matrix elements for the momentum operator and in the third for the holonomy operator. As expected the matrix elements are concentrated at $g = g'$ and simply given by the expectation value of the operator in question times the matrix element of the unit operator so that to leading order in t the expectation value of the monomial is indeed the monomial of the expectation values.

The expectation values of operator monomials are computed in parts two and three by using the overcompleteness of the coherent states. This displays them as particularly useful in deriving a coherent states path integral formulation of the theory [61]. In contrast, in the fourth part we use a different method based on an $SL(2, \mathbb{C})$ operator identity and the so-called moment problem due to Hamburger which we deal with in detail in the second subsection.

3.1.1 The Completeness Relation

Recall from [46] for the case of a general compact Lie group or from [41] for the special case of $G = SU(2)$ that the coherent states ψ_g^t possess the following “reproducing property”

$$(\hat{U}_t \psi)(g) = \langle \psi_{g^*}^t, \psi \rangle_{\mu_H} \quad (3.14)$$

where $g^* = (\bar{g}^T)^{-1}$ is the unique involution on $G^\mathbb{C}$ that preserves G . Here $\psi \in L_2(G, d\mu_H) =: \mathcal{H}$ is an arbitrary state and $\hat{U}_t : \mathcal{H} \mapsto \mathcal{H}^\mathbb{C} = L_2(G^\mathbb{C}, d\nu_t) \cap \text{Hol}(G^\mathbb{C})$ the coherent state transform of [46], that is, the generalization of (3.3) to arbitrary states (heat kernel evolution followed by analytic continuation) mapping the μ_H square integrable functions on G to holomorphic, ν_t square integrable functions on $G^\mathbb{C}$. The measure ν_t of this Bargmann-Segal Hilbert space, defined generally in [46] is computed explicitly in [41] for the case of

$G = SU(2)$ and is chosen such that \hat{U}_t is a unitary operator. Using this unitarity and the reproducing property we compute

$$\begin{aligned} \langle \psi, \psi' \rangle_{\mu_H} &= \langle \hat{U}_t \psi, \hat{U}_t \psi' \rangle_{\nu_t} = \int_{G^{\mathbb{C}}} d\nu_t(g) \overline{(\hat{U}_t \psi)(g)} (\hat{U}_t \psi')(g) \\ &= \int_{G^{\mathbb{C}}} d\nu_t(g) \langle \psi, \psi_{g^*}^t \rangle_{\mu_H} \langle \psi_{g^*}^t, \psi' \rangle_{\mu_H} \end{aligned} \quad (3.15)$$

and using the involution invariance of the measure ν_t which is essentially a Gaussian in p_j we find the *completeness relation*

$$\int_{G^{\mathbb{C}}} d\nu_t(g) |\psi_g^t\rangle \langle \psi_g^t| = 1_{\mathcal{H}} \quad (3.16)$$

Suppose now that we are given an operator monomial $\hat{O} = \hat{O}_1 \dots \hat{O}_n$ where each of the \hat{O}_k , $k = 1, \dots, n < \infty$ stands for one of the elementary operators \hat{h}_{AB}, \hat{p}_j . Then, using (3.16), we can write the expectation value of \hat{O} as

$$\begin{aligned} & \frac{\langle \psi_g^t, \hat{O} \psi_g^t \rangle}{\|\psi_g^t\|^2} \\ &= \frac{1}{\|\psi_g^t\|^2} \int_{G^{\mathbb{C}}} d\nu_t(g_1) \dots \int_{G^{\mathbb{C}}} d\nu_t(g_{n-1}) \prod_{k=1}^n [\langle \psi_{g_{k-1}}^t, \hat{O}_k \psi_{g_k}^t \rangle] \\ &= \int_{G^{\mathbb{C}}} d\Omega(g_1) \dots \int_{G^{\mathbb{C}}} d\Omega(g_{n-1}) \left(\prod_{k=1}^{n-1} [\nu_t(g_k) \|\psi_{g_k}^t\|^2] \right) \left(\prod_{k=1}^n \left[\frac{\langle \psi_{g_{k-1}}^t, \hat{O}_k \psi_{g_k}^t \rangle}{\|\psi_{g_{k-1}}^t\| \|\psi_{g_k}^t\|} \right] \right) \end{aligned} \quad (3.17)$$

where Ω is the Liouville measure on $G^{\mathbb{C}} \cong T^*G$ and we have set $g_0 = g_n = g$. Now we recall from [41] that the quantity

$$j^t(g, g') = \frac{\langle \psi_g^t, \psi_{g'}^t \rangle}{\|\psi_g^t\| \|\psi_{g'}^t\|} \quad (3.18)$$

is exponentially small (at least for $G = SU(2)$) in the sense of a Gaussian needle of width \sqrt{t} unless $g = g'$ (where it equals unity of course) and that the quantity $\nu_t(g) \|\psi_g^t\|^2$ approaches exponentially fast the constant $2/(\pi t^3)$. Thus, it is conceivable that

$$\frac{\langle \psi_{g_{k-1}}^t, \hat{O}_k \psi_{g_k}^t \rangle}{\|\psi_{g_{k-1}}^t\| \|\psi_{g_k}^t\|} \approx \frac{\langle \psi_{g_k}^t, \hat{O}_k \psi_{g_k}^t \rangle}{\|\psi_{g_k}^t\|^2} j^t(g_{k-1}, g_k) \quad (3.19)$$

If that would be the case then the product Liouville measure $d\Omega(g_1) \dots d\Omega(g_{n-1})$ would be essentially supported at $g_1 = \dots = g_{n-1} = g$ and we could pull the expectation values in (3.19) out of the integral (3.17) with g_k replaced by g and the remaining integral would then equal unity, of course. Thus we would have indeed shown that

$$\frac{\langle \psi_g^t, \hat{O} \psi_g^t \rangle}{\|\psi_g^t\|^2} \approx \prod_{k=1}^n \frac{\langle \psi_g^t, \hat{O}_k \psi_g^t \rangle}{\|\psi_g^t\|^2} \quad (3.20)$$

Thus, in order to prove the desired result (3.20) it is sufficient to prove (3.19) together with the precise meaning of \approx . The proof of (3.19) will also be the key ingredient for the derivation of path integrals based on the coherent states ψ_g^t [61].

3.1.2 Matrix Elements for the Momentum Operator

Recall from the beginning of this section that $\hat{p}_j = itX_j/2$ where $X_j(h) = \text{tr}((\tau_j h)^T \partial/\partial h)$ is a basis of right invariant vector fields on G which generate left translations. Thus for any vector $\psi \in \mathcal{H}$ in the domain of \hat{p}_j (say smooth square integrable functions) we have

$$(\hat{p}_j \psi)(h) = \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \psi(e^{s\tau_j} h) \quad (3.21)$$

Since for our coherent states it holds that $\psi_g^t(uh) = \psi_{u^{-1}g}^t(h)$ we have

$$\hat{p}_j \psi_g^t = \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \psi_{e^{-s\tau_j} g}^t \quad (3.22)$$

It follows by explicit computation of the scalar product (see [41] for details) that

$$\langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle = \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \psi_{e^{-s\tau_j} g' \bar{g}^T}^{2t}(1_2) \quad (3.23)$$

Upon defining the complex number z uniquely via (see [41] for details)

$$\cosh(z) = \frac{1}{2} \text{tr}(e^{-s\tau_j} g' \bar{g}^T) = \frac{1}{2} [\text{tr}(g' \bar{g}^T) - \text{str}(\tau_j g' \bar{g}^T)] + O(s^2) \quad (3.24)$$

and using the Weyl character formula for $G = SU(2)$ we obtain with $d_j = 2j + 1$, $j = 0, 1/2, 1, 3/2, \dots$ the spin quantum numbers

$$\begin{aligned} \langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle &= \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \sum_j d_j e^{-tj(j+1)} \frac{\sinh(d_j z)}{\sinh(z)} \\ &= \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/4}}{\sinh(z)} \sum_j d_j e^{-td_j^2/4} \sinh(d_j z) \\ &= \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/4}}{2 \sinh(z)} \sum_n n e^{-tn^2/4} e^{nz} \\ &= \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/4}}{2 \sinh(z) T} \sum_n (nT) e^{-(nT)^2} e^{(nT)\xi} \end{aligned} \quad (3.25)$$

where $n \in \mathbb{Z}$ and we have defined $T = \sqrt{t}/2$, $\xi = z/T$. The Fourier transform $\tilde{f}(k) = \int_{\mathbb{R}} \frac{dx}{2\pi} e^{-ikx} f(x)$ of the function $f(x) := x e^{-x^2} e^{x\xi}$ is given by

$$\tilde{f}(k) = \frac{\xi - ik}{4\sqrt{\pi}} e^{\frac{(\xi - ik)^2}{4}} \quad (3.26)$$

using a contour argument. An appeal to the Poisson summation formula (see, e.g., [62, 41])

$$\sum_n f(nT) = \frac{2\pi}{T} \sum_n \tilde{f}\left(\frac{2\pi n}{T}\right) \quad (3.27)$$

then reveals

$$\langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle = \frac{it}{2} \left(\frac{d}{ds} \right)_{s=0} \frac{\sqrt{\pi} e^{t/4}}{4 \sinh(z) T^3} \sum_n (z - 2\pi i n) e^{\frac{(z - 2\pi i n)^2}{t}} \quad (3.28)$$

By the very same methods we easily obtain

$$\begin{aligned} \|\psi_g^t\|^2 &= \frac{\sqrt{\pi} e^{t/4}}{4 \sinh(p) T^3} \sum_n (p - 2\pi i n) e^{\frac{(p - 2\pi i n)^2}{t}} \\ \|\psi_{g'}^t\|^2 &= \frac{\sqrt{\pi} e^{t/4}}{4 \sinh(p') T^3} \sum_n (p' - 2\pi i n) e^{\frac{(p' - 2\pi i n)^2}{t}} \end{aligned} \quad (3.29)$$

where $\cosh(p) = \text{tr}(g\bar{g}^T)/2$ with the polar decomposition $g = e^{-ip_j\tau_j/2}h$, $p = \sqrt{p_j p_j}$ and similar for g' .

Let now $\cosh(z_0) := \frac{1}{2}\text{tr}(g'\bar{g}'^T)$ and $z = z_0 + \delta$ where δ is of first order in s . Then comparing

$$\cosh(z) = \cosh(z_0) \cosh(\delta) + \sinh(z_0) \sinh(\delta) = \cosh(z_0) + \delta \sinh(z_0) + O(\delta^2) \quad (3.30)$$

with (3.24) we conclude that

$$\delta = -s \frac{\text{tr}(\tau_j g' \bar{g}'^T)}{2 \sinh(z_0)} \quad (3.31)$$

where the sign of $\sinh(z_0)$ follows from the formulas given in [41]. It follows that

$$\left(\frac{d}{ds}\right)_{s=0} = -\frac{\text{tr}(\tau_j g' \bar{g}'^T)}{2 \sinh(z_0)} \left(\frac{d}{dz}\right)_{z=z_0} \quad (3.32)$$

and performing the derivative in (3.28) we end up with

$$\begin{aligned} & \langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle = -\frac{it \text{tr}(\tau_j g' \bar{g}'^T)}{4 \sinh(z_0)} \frac{\sqrt{\pi} e^{t/4}}{4T^3} \times \\ & \times \sum_n e^{\frac{(z_0 - 2\pi in)^2}{t}} \left[2 \frac{(z_0 - 2\pi in)^2}{t \sinh(z_0)} - (z_0 - 2\pi in) \frac{\cosh(z_0)}{\sinh^2(z_0)} + \frac{1}{\sinh(z_0)} \right] \end{aligned} \quad (3.33)$$

Combination with (3.29) results in the exact expression for the momentum matrix element

$$\begin{aligned} \langle \hat{p}_j \rangle_{gg'}^t &:= \frac{\langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle}{\|\psi_g^t\| \|\psi_{g'}^t\|} \\ &= \left[\frac{-iz_0}{2} \frac{\text{tr}(\tau_j g' \bar{g}'^T)}{\sinh(z_0)} \right] \times \\ &\times \frac{\left\{ \frac{t}{2z_0} \sum_n e^{\frac{(z_0 - 2\pi in)^2 - p^2/2 - (p')^2/2}{t}} \left[2 \frac{(z_0 - 2\pi in)^2}{t \sinh(z_0)} - (z_0 - 2\pi in) \frac{\cosh(z_0)}{\sinh^2(z_0)} + \frac{1}{\sinh(z_0)} \right] \right\}}{\left[\sum_n \frac{p - 2\pi in}{\sinh(p)} e^{-\frac{4\pi^2 n^2}{t}} e^{-i\frac{4\pi np}{t}} \right]^{1/2} \left[\sum_n \frac{p' - 2\pi in}{\sinh(p')} e^{-\frac{4\pi^2 n^2}{t}} e^{-i\frac{4\pi np'}{t}} \right]^{1/2}} \end{aligned} \quad (3.34)$$

The arguments $D^t(p)$, $D^t(p')$ of the square roots in the denominator of (3.34) were already estimated in [41] and we will only recall the result here without derivation

$$\frac{p}{\sinh(p)}(1 - K_t) \leq D^t(p) \leq \frac{p}{\sinh(p)}(1 + K_t) \text{ where } K_t = 2 \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \left(1 + \frac{8\pi^2 n}{t}\right) \quad (3.35)$$

vanishes exponentially fast with $t \rightarrow 0$ and similar for $D^t(p')$ with p replaced by p' . The term in the curly brackets of the numerator in (3.34) can be more explicitly written as

$$\begin{aligned} N^t(g, g') &= e^{\frac{z_0^2 - p^2/2 - (p')^2}{t}} \left\{ \left[\frac{z_0}{\sinh(z_0)} - \frac{t \cosh(z_0)}{2 \sinh^2(z_0)} + \frac{1}{z_0 \sinh(z_0)} \right] \right. \\ &+ \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \left[\left(\frac{2z_0^2 - 8\pi^2 n^2}{z_0 \sinh(z_0)} - \frac{t \cosh(z_0)}{\sinh^2(z_0)} + \frac{t}{z_0 \sinh(z_0)} \right) \cos\left(\frac{4\pi n z_0}{t}\right) \right. \\ &\left. \left. + 2\pi n t \left(\frac{\cosh(z_0)}{z_0 \sinh^2(z_0)} - \frac{4}{t \sinh(z_0)} \right) \sin\left(\frac{4\pi n z_0}{t}\right) \right] \right\} \end{aligned} \quad (3.36)$$

which is superficially divergent at the points $z_0 = 0, i\pi$ which reminds us, of course, of the singularity structure of the overlap function in [41] and we will proceed similarly to estimate (3.36). Thus, we will separate the discussion into cases A) and B) respectively, writing $N^t(g, g')$ in terms of z_0 and $z'_0 = z_0 - i\pi$ respectively for $0 \leq \phi \leq (1 - c)\pi$ and

$(1-c)\pi \leq \phi \leq \pi$ respectively where $0 < c < 1$ is a constant. As shown in [41], $z_0 = s + i\phi$ is always uniquely determined with $s \in \mathbb{R}$, $\phi \in [0, \pi]$.

Case A)

Let us pull out a factor of $z_0/\sinh(z_0)$ from (3.36) since at $g = g'$ it will cancel against the $p/\sinh(p)$ coming from $D^t(p)$ and separate terms into those which are regular and irregular respectively at $z_0 = 0$, resulting in

$$\begin{aligned} N^t(g, g') &= e^{\frac{z_0^2 - p^2/2 - (p')^2}{t}} \frac{z_0}{\sinh(z_0)} \left\{ [1] + \left[\frac{t}{2} \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} \right] \right. \\ &+ \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \left([2 \cos(\frac{4\pi n z_0}{t})] + \left[t \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} \cos(\frac{4\pi n z_0}{t}) \right] - [8\pi n \sin(\frac{4\pi n z_0}{t})/z_0] \right. \\ &\left. \left. + 2\pi n t \left[\frac{\cosh(z_0)}{z_0^2 \sinh(z_0)} \sin(\frac{4\pi n z_0}{t}) - \frac{4\pi n}{t z_0^2} \cos(\frac{4\pi n z_0}{t}) \right] \right) \right\} \end{aligned} \quad (3.37)$$

and obviously all terms in the square brackets, except for the last one, are regular at $z_0 = 0$. However, expanding numerator and denominator to second order in z_0 we see

$$\begin{aligned} &\frac{\cosh(z_0)}{z_0^2 \sinh(z_0)} \sin(\frac{4\pi n z_0}{t}) - \frac{4\pi n}{t z_0^2} \cos(\frac{4\pi n z_0}{t}) \\ &= \frac{1}{z_0 \sinh(z_0)} \left[\cosh(z_0) \frac{\sin(\frac{4\pi n z_0}{t})}{z_0} - \frac{4\pi n \sinh(z_0)}{t z_0} \cos(\frac{4\pi n z_0}{t}) \right] \\ &= \frac{1}{z_0^2 (1 + O(z_0^2))} \left[\frac{4\pi n}{t} (1 + O(z_0^2)) - \frac{4\pi n}{t} (1 + O(z_0^2)) \right] = O(z_0^2) \end{aligned} \quad (3.38)$$

so that (3.38) even vanishes at $z_0 = 0$.

We want to put bounds on all those terms in the square brackets of (3.37) for $0 \leq \phi \leq (1-c)\pi$ except for the first one and, in particular, estimate the series. To that end we write (3.37) in a yet more suggestive form

$$\begin{aligned} N^t(g, g') &= e^{\frac{z_0^2 - p^2/2 - (p')^2/2}{t}} \frac{z_0}{\sinh(z_0)} \times \\ &\times \left\{ [1] + \left[\frac{t}{2} \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} \right] \left[1 + 2 \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \cos(\frac{4\pi n z_0}{t}) \right] \right. \\ &+ 2 \left[\sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \left(\cos(\frac{4\pi n z_0}{t}) - \frac{(4\pi n)^2 \sin(\frac{4\pi n z_0}{t})}{t} \right) \right] \\ &\left. + 2\pi t \left[\sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} n \left(\frac{\cosh(z_0)}{z_0 \sinh(z_0)} \frac{\sin(\frac{4\pi n z_0}{t})}{z_0} - \frac{4\pi n}{t z_0^2} \cos(\frac{4\pi n z_0}{t}) \right) \right] \right\} \\ &=: e^{\frac{z_0^2 - p^2/2 - (p')^2/2}{t}} \frac{z_0}{\sinh(z_0)} \times \\ &\times \left\{ 1 + \frac{t}{2} \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} I_1 + 2I_2 + 2\pi t I_3 \right\} \end{aligned} \quad (3.39)$$

and it remains to estimate the sums I_1, I_2, I_3 as well as the expression

$$I := \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} \quad (3.40)$$

Notice that for the terms proportional to t in (3.39) it will be sufficient to estimate them by a function integrable against the Gaussian prefactor of (3.39).

Focussing first on (3.40) we first prove two elementary lemmas.

Lemma 3.1 For any $z = s + i\phi \in \mathbb{C}$ such that $0 \leq \phi \leq (1 - c)\pi$ for some $0 < c < 1$ we have

$$\frac{s^2}{\sinh^2(s)} \leq \left| \frac{z}{\sinh(z)} \right|^2 \leq \frac{\phi^2}{\sin^2(\phi)} \leq \left[\frac{\pi(1-c)}{\sin(\pi(1-c))} \right]^2 =: k_c^2 \quad (3.41)$$

$$\left| \frac{z}{\sinh(z)} \right|^2 \leq \frac{\phi^2}{\sin^2(\phi)} \frac{s^2}{\sinh^2(s)} \leq k_c^2 \frac{s^2}{\sinh^2(s)} \quad (3.42)$$

Proof of Lemma 3.1 :

Notice the identity

$$\left| \frac{z}{\sinh(z)} \right|^2 = \frac{s^2 + \phi^2}{\sinh^2(s) \cos^2(\phi) + \cosh^2(s) \sin^2(\phi)} = \frac{s^2 + \phi^2}{\sinh^2(s) + \sin^2(\phi)} \quad (3.43)$$

Both the lower and upper bounds in (3.41) turn out to be equivalent with the inequality

$$\left(\frac{\sinh(s)}{s} \right)^2 \geq \left(\frac{\sin(\phi)}{\phi} \right)^2 \quad (3.44)$$

which is true as the left hand side and right hand side respectively both take its minimum and maximum respectively at $s = \phi = 0$, in fact, the left hand side and right hand side respectively are strictly increasing and decreasing functions respectively.

Inequality (3.42) in turn can be transformed into the equivalent form

$$\frac{1}{s^2} - \frac{1}{\sinh^2(s)} \leq \frac{1}{\sin^2(\phi)} - \frac{1}{\phi^2} \quad (3.45)$$

Both the left and the right hand side of this inequality are always positive in the range considered and in fact the left hand side approaches 0 as $s \rightarrow \infty$ while the right hand side approaches $+\infty$ as $\phi \rightarrow \pi$. At $\phi = s = 0$ both sides equal $1/3$. We will in fact prove that

$$\frac{1}{s^2} - \frac{1}{\sinh^2(s)} \leq \frac{1}{3} \leq \frac{1}{\sin^2(\phi)} - \frac{1}{\phi^2} \quad (3.46)$$

Consider first the left hand side. This can be written in the equivalent form

$$(3 - s^2) \sinh^2(s) \leq 3s^2 \quad (3.47)$$

which is obviously true for $s^2 \geq 3$ so that we may restrict examination to $s^2 < 3$. In that case we may write (3.47) in the equivalent form

$$\frac{\cosh(2s) - 1}{2s^2} \leq \frac{1}{1 - s^2/3} \quad (3.48)$$

As $s^2/3 < 1$ the right hand side can be expanded into a geometric series. Introducing $x = 2s$ and employing the Taylor series for cosh we can write (3.48) as

$$\sum_{n=0}^{\infty} x^{2n} \left[\frac{1}{12^n} - \frac{2}{(2(n+1))!} \right] \geq 0 \quad (3.49)$$

and it will be sufficient to establish non-negativity of every coefficient. Using the basic estimate $\ln(n!) \geq n(\ln(n) - 1) + 1$ valid for $n \geq 1$ it is easy to see that this is indeed the case for $n > 4$ while for the cases $n = 0, 1, 2, 3, 4$ this can be checked by direct computation.

Turning to the right hand side of (3.46) we can write it in the equivalent form

$$3\phi^2 \geq \sin^2(\phi)[3 + \phi^2] \quad (3.50)$$

which due to $\sin^2(\phi) \leq 1$ is certainly true for $\phi^2 \geq 3/2$ so that we can focus attention on the case $\phi^2/3 < 3/2$. Writing (3.50) in the equivalent form

$$\frac{1 - \cos(2\phi)}{2\phi^2} \leq \frac{1}{1 - (-\phi^2/3)} \quad (3.51)$$

exploiting that $0 \leq \phi^2/3 < 1/2$ lies in the radius of convergence of the geometric series, introducing $0 \leq y = (2\phi)^2/6 < 1$ and

$$b_n := \frac{1}{2^n} - \frac{2 \cdot 6^n}{(2(n+1))!} \quad (3.52)$$

we may write (3.51) in the form

$$\sum_{n=0}^{\infty} (-1)^n y^n b_n = \sum_{n=0}^{\infty} [y^{2n} b_{2n} - y^{2n+1} b_{2n+1}] \geq 0 \quad (3.53)$$

Since $0 \leq y < 1$ this will be the case if $b_{2n} \geq b_{2n+1}$ for all $n \geq 0$. As one can check, $b_0 = b_1 = 0$ and in (3.49) we have already seen that $b_n > 0$ for $n \geq 2$. Thus, it is enough to prove $b_{2n} \geq b_{2n+1}$ for $n \geq 1$. In fact, we will prove more, namely that b_n is strictly decreasing for $n \geq 2$. This turns out to be equivalent with $(2(n+1))! - 2 \cdot 12^n > 1$ for all $n \geq 2$ which in turn would follow from $(2(n+1))! > (2 + \frac{1}{144})12^n$. The latter condition can be demonstrated to be true by methods similar to those outlined in (3.49). \square

Lemma 3.2 *Let*

$$S(z) := \frac{\sinh(z) - z}{z^2 \sinh(z)}, \quad C(z) := \frac{\cosh(z) - 1}{z \sinh(z)}, \quad k'_c := \sqrt{1 + k_c \cosh(\pi(1 - c))} \quad (3.54)$$

Then for any $z = s + i\phi \in \mathbb{C}$, $0 \leq \phi \leq (1 - c)\pi$ it holds that

$$|S(z)| \leq \sqrt{2}k'_c \text{ and } |C(z)| \leq \sqrt{2}k'_c \quad (3.55)$$

Proof of Lemma 3.2 :

Using hyperbolic and trigonometric identities we derive

$$\begin{aligned} |S(z)|^2 &= \frac{(\sinh(s) \cos(\phi) - s)^2 + (\cosh(s) \sin(\phi) - \phi)^2}{|z|^4 (\cosh^2(s) - \cos^2(\phi))} \\ &= \frac{(\frac{s^2}{|z|^2} \frac{\sinh(s)-s}{s^2} \cos(\phi) + \phi \frac{s\phi}{|z|^2} \frac{\cos(\phi)-1}{\phi^2})^2 + (\frac{s\phi}{|z|^2} \frac{\cosh(s)-1}{s} \frac{\sin(\phi)}{\phi} + \frac{\phi^2}{|z|^2} \phi \frac{\sin(\phi)-\phi}{\phi^3})^2}{\cosh^2(s) - \cos^2(\phi)} \\ &\leq \frac{(|\frac{\sinh(s)-s}{s^2}| |\cos(\phi)| + \phi |\frac{\cos(\phi)-1}{\phi^2}|)^2 + (|\frac{\cosh(s)-1}{s}| |\frac{\sin(\phi)}{\phi}| + \phi |\frac{\sin(\phi)-\phi}{\phi^3}|)^2}{\cosh^2(s) - \cos^2(\phi)} \\ &\leq \frac{(|\frac{\sinh(s)-s}{s^2}| + \phi |\frac{\cos(\phi)-1}{\phi^2}|)^2 + (|\frac{\cosh(s)-1}{s}| + \phi |\frac{\sin(\phi)-\phi}{\phi^3}|)^2}{\cosh^2(s) - \cos^2(\phi)} \\ &\leq (|\frac{\sinh(s)-s}{s^2 \sinh(s)}| + |\frac{\phi}{\sin(\phi)}| |\frac{\cos(\phi)-1}{\phi^2}|)^2 + (|\frac{\cosh(s)-1}{s \sinh(s)}| + |\frac{\phi}{\sin(\phi)}| |\frac{\sin(\phi)-\phi}{\phi^3}|)^2 \\ &\leq (|S(s)| + k_c |\frac{\cos(\phi)-1}{\phi^2}|)^2 + (|C(s)| + k_c |\frac{\sin(\phi)-\phi}{\phi^3}|)^2 \end{aligned} \quad (3.56)$$

where in the first inequality we used $|s|, |\phi| \leq |z|$, in the second that $|\cos(\phi)|, |\sin(\phi)/\phi| \leq 1$, in the third that $\cosh^2(s) - \cos^2(\phi) \geq \sinh^2(s), \sin^2(\phi)$ and in the fourth we used Lemma 3.1 and again the definition of $S(z), C(z)$.

Proceeding similarly with $C(z)$ we arrive at

$$\begin{aligned}
|C(z)|^2 &= \frac{(\cosh(s) \cos(\phi) - 1)^2 + (\sinh(s) \sin(\phi))^2}{|z|^2(\cosh^2(s) - \cos^2(\phi))} \\
&= \frac{(s \frac{\cosh(s)-1}{s} \cos(\phi) + \phi^2 \frac{\cos(\phi)-1}{\phi^2})^2 + (\phi \sinh(s) \frac{\sin(\phi)}{\phi})^2}{|z|^2(\cosh^2(s) - \cos^2(\phi))} \\
&\leq \frac{(|\frac{\cosh(s)-1}{s}| |\cos(\phi)| + \phi |\frac{\cos(\phi)-1}{\phi^2}|)^2 + (\sinh(s) |\frac{\sin(\phi)}{\phi}|)^2}{(\cosh^2(s) - \cos^2(\phi))} \\
&\leq \frac{(|\frac{\cosh(s)-1}{s}| + \phi |\frac{\cos(\phi)-1}{\phi^2}|)^2 + \sinh^2(s)}{(\cosh^2(s) - \cos^2(\phi))} \\
&\leq (|\frac{\cosh(s)-1}{s \sinh(s)}| + |\frac{\phi}{\sin(\phi)}| |\frac{\cos(\phi)-1}{\phi^2}|)^2 + 1 \\
&\leq (|C(s)| + k_c |\frac{\cos(\phi)-1}{\phi^2}|)^2 + 1
\end{aligned} \tag{3.57}$$

Using the Taylor series expansion of the trigonometric functions we easily obtain

$$\begin{aligned}
|\frac{\cos(\phi)-1}{\phi^2}| &= |\sum_{n=1}^{\infty} \frac{(-1)^n \phi^{2(n-1)}}{(2n)!}| = |\sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+2)!}| \leq \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(2(n+1))!} \\
&\leq \cosh(\phi) \leq \cosh(\pi(1-c)) \\
|\frac{\sin(\phi)-\phi}{\phi^3}| &= |\sum_{n=1}^{\infty} \frac{(-1)^n \phi^{2(n-1)}}{(2n+1)!}| = |\sum_{n=0}^{\infty} \frac{(-1)^n \phi^{2n}}{(2n+3)!}| \leq \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(2n+3)!} \leq \cosh(\phi) \\
&\leq \cosh(\pi(1-c))
\end{aligned} \tag{3.58}$$

Next notice that $S(s) = S(|s|)$, $C(s) = C(|s|)$ so that we have reduced our estimate for $|S(z)|, |C(z)|$ to that for real non-negative arguments. Finally, using the Taylor series expression for the hyperbolic functions we find

$$\begin{aligned}
|\frac{\cosh(s)-1}{s}| &= |\sum_{n=1}^{\infty} \frac{s^{2n-1}}{(2n)!}| = |\sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+2)!}| \leq \sinh(|s|) \\
|\frac{\sinh(s)-s}{s^2}| &= |\sum_{n=1}^{\infty} \frac{\phi^{2n-1}}{(2n+1)!}| = |\sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+3)!}| \leq \sinh(|s|)
\end{aligned} \tag{3.59}$$

so that in fact $|S(s)|, |C(s)| \leq 1$. Together with the trivial inequality $1 \leq k_c$ we thus obtain indeed $|C(z)|^2, |S(z)|^2 \leq 2[1 + k_c \cosh(\pi(1-c))]^2 = 2(k'_c)^2$.

□

Now we can give a bound on I .

Lemma 3.3 *For any $z_0 = s + i\phi \in \mathbb{C}$ such that $0 \leq \phi \leq (1-c)\pi$ for some $0 < c < 1$ we have*

$$|I| \leq 2k'_c \tag{3.60}$$

Proof of Lemma 3.3 :

This follows immediately from the identity

$$I = S(z) - C(z) \tag{3.61}$$

and lemma 3.2.

□

Next consider the following expression that appears in I_3

$$J := \frac{\cosh(z_0)}{z_0 \sinh(z_0)} \frac{\sin(\frac{4\pi n z_0}{t})}{z_0} - \frac{4\pi n}{t z_0^2} \cos(\frac{4\pi n z_0}{t}) \quad (3.62)$$

which we must estimate in such a way that finally s and n do not appear in the combination ns inside a hyperbolic function as otherwise we must worry about convergence of the series I_3 as $|s|$ becomes arbitrarily large in the integrals we are considering. At the same time we must bound the superficial singularity at $z_0 = 0$. To that end we introduce $z'_0 = 4\pi n z_0/t$ and notice the identity, recalling the definition (3.40)

$$J = \frac{4\pi n}{t} \left\{ -\frac{\sin(z'_0)}{z'_0} I + \left(\frac{4\pi n}{t}\right)^2 \left[\frac{\sin(z'_0) - z'_0}{(z'_0)^3} - \frac{\cos(z'_0) - 1}{(z'_0)^2} \right] \right\} \quad (3.63)$$

and the task is to estimate the three terms of the form $(\cos(z) - 1)/z^2$, $\sin(z)/z$, $(\sin(z) - z)/z^3$ for arbitrary $z = x + iy \in \mathbb{C}$. The inequality

$$\left| \frac{\sin(z)}{z} \right| \leq 2 \cosh(y) \quad (3.64)$$

is the content of lemma 4.1 of [41]. The remaining two estimates are the hardest ones and we have therefore devoted the subsequent lemma to them.

Lemma 3.4 *Let for any complex number z*

$$c(z) := \frac{\cos(z) - 1}{z^2} \text{ and } s(z) := \frac{\sin(z) - z}{z^3} \quad (3.65)$$

Then

$$|c(z)| \leq 4 \cosh(\Im(z)) \text{ and } |s(z)| \leq 4 \cosh(\Im(z)) \quad (3.66)$$

Proof of Lemma 3.4 :

i)

Splitting $z = x + iy$ we have

$$\begin{aligned} \left| \frac{\cos(z) - 1}{z^2} \right| &= \left| \frac{[\cos(x) \cosh(y) - 1] - i[\sin(x) \sinh(y)]}{z^2} \right| \\ &\leq \left| \frac{xy}{z^2} \right| \left| \frac{\sin(x)}{x} \right| \left| \frac{\sinh(y)}{y} \right| + \left| \frac{[\cos(x) - 1][\cosh(y) - 1] + [\cos(x) - 1] + [\cosh(y) - 1]}{z^2} \right| \\ &\leq \sinh(|y|) + \left| \frac{xy}{z^2} \right| \left| \frac{\cosh(y) - 1}{y} \right| \left| \frac{\cos(x) - 1}{x} \right| + \left| \frac{x^2}{z^2} \right| \left| \frac{\cos(x) - 1}{x^2} \right| + \left| \frac{y^2}{z^2} \right| \left| \frac{\cosh(y) - 1}{y^2} \right| \\ &\leq \sinh(|y|) + \left| \frac{\cosh(y) - 1}{y} \right| \left| \frac{\cos(x) - 1}{x} \right| + \left| \frac{\cos(x) - 1}{x^2} \right| + \left| \frac{\cosh(y) - 1}{y^2} \right| \end{aligned} \quad (3.67)$$

using $|x/z|, |y/z| \leq 1$. It is easy to see that $|\cosh(y) - 1|/y^2 \leq \cosh(y)$ by using the Taylor series of $\cosh(y)$, see e.g. [41]. By similar methods it is easy to establish that $|\frac{\cosh(y)-1}{y}| \leq \sinh(|y|)$. Furthermore, we claim that $|\frac{\cos(x)-1}{x}|, |\frac{\cos(x)-1}{x^2}| \leq 1$.

To see the former, notice that $|\frac{\cos(x)-1}{x}| = |\frac{\cos(|x|)-1}{|x|}|$ so it will be sufficient to demonstrate this for $x \geq 0$. Now for $x \geq 0$ the inequality $|\frac{\cos(x)-1}{x}| \leq 1$ is equivalent with $1 - x \leq \cos(x) \leq 1 + x$, so we claim that $f_{\pm}(x) = x \pm (1 - \cos(x))$ are not negative functions for $x \geq 0$. But $f'_{\pm}(x) = 1 \pm \sin(x) \geq 0$ so that f_{\pm} is never decreasing and takes its minimum at $x = 0$ where $f_{\pm}(0) = 0$.

To see the latter, notice that $|\frac{\cos(x)-1}{x^2}| = |\frac{\cos(|x|)-1}{|x|^2}|$ so it will be sufficient to demonstrate this for $x \geq 0$ as well. Now for $x \geq 0$ the inequality $|\frac{\cos(x)-1}{x^2}| \leq 1$ is equivalent with $1 - x^2 \leq \cos(x) \leq 1 + x^2$, so we claim that $f_{\pm}(x) = x^2 \pm (1 - \cos(x))$ are not negative functions for $x \geq 0$. But $g_{\pm} = f'_{\pm}(x) = 2x \pm \sin(x)$ and $g'_{\pm}(x) = 2 \pm \cos(x) > 0$. Thus, g_{\pm} is strictly increasing, its minimum at $x = 0$ being $g_{\pm}(0) = 0$. Thus f_{\pm} is never decreasing and takes its minimum at $x = 0$ where $f_{\pm}(0) = 0$. It follows that

$$|\frac{\cos(z)-1}{z^2}| \leq \sinh(|y|) + \sinh(|y|) + 1 + \cosh(y) \leq 4 \cosh(y) \quad (3.68)$$

since $1, \sinh(|y|) \leq \cosh(y)$.

ii)

Proceeding similarly as in i) we have

$$\begin{aligned} & \left| \frac{\sin(z) - z}{z^3} \right| = \left| \frac{1}{z^3} ([\sin(x) \cosh(y) - x] + i[\cos(x) \sinh(y) - y]) \right| \\ & \leq \frac{1}{|z^3|} \left| [xy^2 \frac{\sin(x)}{x} \frac{\cosh(y)-1}{y^2}] + [x^3 \frac{\sin(x)-x}{x^3}] + [x^2 y \frac{\cos(x)-1}{x^2} \frac{\sinh(y)}{y}] + [y^3 \frac{\sinh(y)-y}{y^3}] \right| \\ & \leq \left| \frac{\sin(x)}{x} \right| \left| \frac{\cosh(y)-1}{y^2} \right| + \left| \frac{\sin(x)-x}{x^3} \right| + \left| \frac{\cos(x)-1}{x^2} \right| \left| \frac{\sinh(y)}{y} \right| + \left| \frac{\sinh(y)-y}{y^3} \right| \\ & \leq \cosh(y) + \left| \frac{\sin(x)-x}{x^3} \right| + \cosh(y) + \cosh(y) \end{aligned} \quad (3.69)$$

where we have made use of properties already demonstrated in i) and the Taylor series of $\sinh(y)$. We claim that $|\frac{\sin(x)-x}{x^3}| = |\frac{\sin(|x|)-|x|}{|x|^3}| \leq 1$ and it is sufficient to prove this for $x \geq 0$. That statement is for $x \geq 0$ equivalent to $f(x) = x^3 + \sin(x) - x \geq 0$ because $\sin(x) \leq x$ for $x \geq 0$. We have $g(x) = f'(x) = 3x^2 + \cos(x) - 1, h(x) = g'(x) = 6x - \sin(x), h'(x) = 6 - \cos(x) > 0$. Since $f(0) = g(0) = h(0) = 0$ we see that h is an increasing function whence $h(x) \geq 0$ from which follows that g is an increasing function whence $g(x) \geq 0$ from which follows that f is an increasing function whence $f(x) \geq 0$ as claimed. It follows that

$$\left| \frac{\frac{\sin(z)}{z} - 1}{z^2} \right| \leq 1 + 3 \cosh(y) \leq 4 \cosh(y) \quad (3.70)$$

as claimed.

□

Collecting all the estimates we can now write the estimate for the modulus of (3.62) in the desired form, defining $y' = 4\pi n\phi/t$,

$$|J| \leq \frac{4\pi n}{t} [2 \cosh(y') |I| + 8 (\frac{4\pi n}{t})^2 \cosh(y')] \leq \frac{32\pi n}{t} [k'_c + (\frac{4\pi n}{t})^2] \cosh(y') \quad (3.71)$$

which now enables us to estimate the various series I_1, I_2, I_3 . We will do this one by one. I_1 :

The elementary estimate $|\cos(z)| \leq |\cosh(y)| + |\sinh(y)| = e^{|y|}$ applied to I_1 reveals

$$\begin{aligned} |I_1| & \leq 1 + 2 \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} e^{\frac{4\pi^2 n(1-c)}{t}} \\ & \leq 1 + 2e^{-\frac{4\pi^2 c}{t}} \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 (n^2-n)}{t}} \\ & \leq 1 + 2e^{-\frac{4\pi^2 c}{t}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} =: 1 + k_t \end{aligned} \quad (3.72)$$

where in the last step we have made use of the inequality $(n-1)^2 \leq n^2 - n$ valid for $n \geq 1$. The constant k_t is independent of g, g' and vanishes exponentially fast with $t \rightarrow 0$ for any $c > 0$.

I_2 :

Using again $|\cos(z)| \leq e^{|y|}$, $|\sin(z)/z| \leq 2 \cosh(y) \leq 2e^{|y|}$ we easily find with $|\phi| \leq (1-c)\pi$

$$\begin{aligned} |I_2| &\leq \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} e^{\frac{4\pi^2 n(1-c)}{t}} \left(1 + \frac{32\pi^2 n^2}{t}\right) \\ &\leq e^{-\frac{4\pi^2 c}{t}} \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 (n^2-n)}{t}} \left(1 + \frac{32\pi^2 n^2}{t}\right) \\ &\leq e^{-\frac{4\pi^2 c}{t}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \left(1 + \frac{32\pi^2 (n+1)^2}{t}\right) =: k'_t \end{aligned} \quad (3.73)$$

where again k'_t is a constant, approaching zero exponentially fast with $t \rightarrow 0$ for any $c > 0$.

I_3 :

Invoking our estimate (3.71) for the quantity J (3.62) that appears in I_3 we obtain

$$\begin{aligned} |I_3| &\leq \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} e^{\frac{4\pi^2 n(1-c)}{t}} \frac{32\pi n^2}{t} [k'_c + (\frac{4\pi n}{t})^2] \\ &\leq 32\pi e^{-\frac{4\pi^2 c}{t}} \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 (n^2-n)}{t}} \frac{n^2}{t} [k'_c + (\frac{4\pi n}{t})^2] \\ &\leq 32\pi e^{-\frac{4\pi^2 c}{t}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} \frac{(n+1)^2}{t} [k'_c + (\frac{4\pi (n+1)}{t})^2] =: \tilde{k}_t \end{aligned} \quad (3.74)$$

where again \tilde{k}_t is a constant independent of both g, g' exponentially vanishing with $t \rightarrow 0$, for any $c > 0$.

Let us now define

$$\begin{aligned} \Delta < \hat{p}_j >_{gg'}^t &:= < \hat{p}_j >_{gg'}^t - \frac{[\frac{-i}{2} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{\sinh(z_0)} z_0] e^{\frac{z_0^2 - p^2/2 - (p')^2/2}{t}} \frac{z_0}{\sinh(z_0)}}{\sqrt{D^t(p) D^t(p')}} \\ &= \frac{[\frac{-i}{2} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{\sinh(z_0)} z_0] e^{\frac{z_0^2 - p^2/2 - (p')^2/2}{t}} \frac{z_0}{\sinh(z_0)} \left\{ \frac{t}{2} \frac{\frac{\sinh(z_0)}{z_0} - \cosh(z_0)}{z_0 \sinh(z_0)} I_1 + 2I_2 + 2\pi t I_3 \right\}}{\sqrt{D^t(p) D^t(p')}} \end{aligned} \quad (3.75)$$

Recall now the relation between the various objects $s, \phi, \tilde{p}, \tilde{\theta}, \tilde{\alpha}$ from section 4.2 of [41]. Basically, one writes $g = Hh, g' = H'h'$ in polar decomposed form and defines the polar decomposition $HH' = \tilde{H}u$ as well as $\tilde{h} = h^{-1}h'u^{-1}$. Then $\tilde{H} = \exp(-i\tilde{p}_j \tau_j/2)$, $\tilde{h} = \exp(\tilde{\theta}_j \tau_j)$ and $\cosh(s) \cos(\phi) = \cosh(\tilde{p}/2) \cos(\tilde{\theta})$ and $\sinh(s) \sin(\phi) = \sinh(\tilde{p}/2) \sin(\tilde{\theta}) \cos(\tilde{\alpha})$ with $\cos(\tilde{\alpha}) = \tilde{p}_j \tilde{\theta}_j / (\tilde{p} \tilde{\theta})$.

Writing $g' \bar{g}^T = e^{-i\tau_j z_0^j}$ we have

$$\left[\frac{-i}{2} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{\sinh(z_0)} z_0 \right] = z_0^j \quad (3.76)$$

and are interested in the relation between z_0^j and $z_0 = s + i\phi$. By definition we have $\cosh(z_0) = \text{tr}(g' \bar{g}^T)/2$ which reveals that $z_0^2 = \sum_j (z_0^j)^2$, however, it is not true that $|z_0|^2 \geq \sum_j |z_0^j|^2$. In order to estimate the integral over $\Delta < \hat{p}_j >_{g,g'}$ we thus have to prove one more relation.

Lemma 3.5 For $z_0 = s + i\phi$, $0 \leq \phi \leq (1 - c)\pi$ and $\tilde{g} := g' \bar{g}^T = e^{-i\tau_j z_0^j}$ with $z_0^2 = z_0^j z_0^j$ we have

$$|z_0^j|^2 \leq [k_c \frac{s}{\sinh(s)} \sqrt{\cosh(\tilde{p})}]^2 \leq [2k_c \frac{s}{\sinh(s)} \cosh(\frac{p+p'}{2})]^2 \quad (3.77)$$

for any $j = 1, 2, 3$.

Proof of Lemma 3.5 :

Using the $SL(2, \mathbb{C})$ “Fierz identity” $\text{tr}(M\tau_j)\tau_j = \text{tr}(M) - 2M$ valid for any 2×2 matrix M we find from (3.76) that

$$\begin{aligned} \sum_j |z_0^j|^2 &= -\frac{1}{4} \left| \frac{z_0}{\sinh(z_0)} \right|^2 \text{tr}(\tau_j \tilde{g}) \text{tr}(\tau_j \tilde{g}^T) \\ &= -\frac{1}{4} \left| \frac{z_0}{\sinh(z_0)} \right|^2 (|\text{tr}(\tilde{g})|^2 - 2\text{tr}(\tilde{g}\tilde{g}^T)) \\ &\leq \frac{1}{2} \left| \frac{z_0}{\sinh(z_0)} \right|^2 \text{tr}(\tilde{H}^2) = \left| \frac{z_0}{\sinh(z_0)} \right|^2 \cosh(\tilde{p}) \end{aligned} \quad (3.78)$$

Now recall from [41] that

$$\cosh(\tilde{p}) = (1 + r) \cosh^2(\frac{p+p'}{2}) + (1 - r) \cosh^2(\frac{p-p'}{2}) - 1 \quad (3.79)$$

where $r = p_j p'_j / (pp') \in [-1, 1]$. Combining (3.79) and (3.78) yields (notice that $p + p' > |p - p'|$, $|r| \leq 1$)

$$\sum_j |z_0^j|^2 \leq 4 \left| \frac{z_0}{\sinh(z_0)} \right|^2 \cosh^2(\frac{p+p'}{2}) \leq [2k_c \frac{s}{\sinh(s)} \cosh(\frac{p+p'}{2})]^2 \quad (3.80)$$

□

Next, by estimates established in [41] we have

$$\Delta^2(\vec{p}, \vec{p}') := p^2 + (p')^2 - \frac{\tilde{p}^2}{2} \geq 0 \text{ and } \delta^2(g, g') := \tilde{p}^2/4 - s^2 + \phi^2 - \tilde{\theta}^2 \geq 0 \quad (3.81)$$

where $\Delta = 0$ if and only if $\vec{p} = \vec{p}'$ and $\delta = 0$ if either a) $\tilde{\alpha} = 0, \pi$ and $\tilde{p}, \tilde{\theta}$ are arbitrary or b) $\tilde{\alpha}$ is arbitrary and one or both of $\tilde{p} = 0; \tilde{\theta} = 0, \pi$ hold. In both of the cases a), b) we have $|s| = \tilde{p}/2, \phi = \tilde{\theta}$. It follows that

$$\Re(z_0^2 - p^2/2 - (p')^2/2) = -\left\{ \left[\frac{p^2}{2} + \frac{(p')^2}{2} - \frac{\tilde{p}^2}{4} \right] + \tilde{\theta}^2 + [-s^2 + \phi^2 - \tilde{\theta}^2 + \frac{\tilde{p}^2}{4}] \right\} = -[\Delta^2/2 + \delta^2 + \tilde{\theta}^2] \quad (3.82)$$

Combining the estimates (3.35) for $D^t(p)$, (3.60) for I , (3.72), (3.73) and (3.74) respectively for I_1, I_2, I_3 respectively, (3.77) for $|z_0^j|$, (3.41) for $f_c(s)$ and (3.82) for the Gaussian prefactor of $N^t(g, g')$ we conclude

$$\begin{aligned} |\Delta < \hat{p}_j >_{gg'}^t| &\leq |z_0^j| e^{\frac{\Re(z_0^2 - p^2/2 - (p')^2/2)}{t}} \left| \frac{z_0}{\sinh(z_0)} \right| \frac{\left\{ \frac{t}{2} |I| |I_1| + 2|I_2| + 2\pi t |I_3| \right\}}{\sqrt{D^t(p) D^t(p')}} \\ &\leq [2k_c \frac{s}{\sinh(s)} \cosh(\frac{p+p'}{2})] [e^{-\frac{\Delta^2/2 + \delta^2 + \tilde{\theta}^2}{t}}] \left[\frac{k_c \frac{s}{\sinh(s)}}{(1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \right] \times \\ &\times \{ [tk'_c(1 + k_t)] + [2k'_t] + [2\pi t \tilde{k}_t] \} \\ &= 2k_c^2 [e^{-\frac{\Delta^2/2 + \tilde{\theta}^2}{t}}] [e^{-\delta^2/t} \frac{s^2}{\sqrt{pp'}} \frac{\cosh(\frac{p+p'}{2}) \sqrt{\sinh(p) \sinh(p')}}{\sinh^2(s)}] \left[\frac{tk'_c(1 + k_t) + 2k'_t + 2\pi t \tilde{k}_t}{1 - K_t} \right] \end{aligned} \quad (3.83)$$

Let us discuss this result. The last line of (3.83) consists of three factors corresponding to the three square brackets. The first bracket contains a Gaussian with peak of width of order \sqrt{t} at $g = g'$. The third bracket is of the form $t(1 + K'_t(c))$ where $K'_t(c)$ is exponentially vanishing with t for any $0 < c < 1$. These two brackets are expected from the harmonic oscillator. The remaining second bracket

$$B_2 := e^{-\delta^2/t} \frac{s^2}{\sqrt{pp'}} \frac{\cosh(\frac{p+p'}{2}) \sqrt{\sinh(p) \sinh(p')}}{\sinh^2(s)} \quad (3.84)$$

is unexpected, it is not manifestly bounded from above and thus the integral over g' is not obviously converging. The subsequent paragraph will be devoted to the behaviour of that term.

Since (3.84) is manifestly regular at $p' = 0$, convergence problems can arise only for $p' \rightarrow \infty$. Formula (3.79) implies that then also $\tilde{p} \rightarrow \infty$, no matter which values p, r take, actually $\tilde{p} \rightarrow p'$ as $p' \rightarrow \infty$ for fixed p, r . By estimate (3.81) we then see that either a) $\delta \rightarrow \infty$ or b) δ stays bounded from above. In the first case s must stay bounded or grows slower than $\tilde{p}/2$ while in the latter case $s \rightarrow \tilde{p}/2 \rightarrow p/2'$. Consider first case a). Then for large p' the Gaussian $e^{-\delta^2/t}$ in (3.84) certainly wins over the remaining factor and B_2 decays exponentially fast. Next consider case b). In that case B_2 grows as $\sqrt{p'}^3$ at large p' . Thus, altogether we have shown that B_2 grows no worse than polynomially as $p' \rightarrow \infty$. Notice that s stays bounded if and only if $\theta = \tilde{\alpha} = \pi/2$ which defines a set of Ω measure zero. In that case in fact $s = 0, \phi = \pi/2$, therefore $z_0 = i\pi/2$ whence $\tilde{g} = \tau_j \tilde{\theta}_j \cosh(\tilde{p}/2)/\tilde{\theta}$ and $\sum_j |z_0^j/z_0|^2 = \cosh^2(\tilde{p}/2)$.

We conclude that in the range $0 \leq \phi \leq (1-c)\pi$ $\Delta < \hat{p}_j >_{gg'}^t$ can be bounded by a function of g, g' which is of the form of a Gaussian with peak at $g = g'$ times a function of g, g' bounded by a polynomial in \vec{p}, \vec{p}' times t times a constant that approaches unity exponentially fast. Thus the integral of that function with respect to g' exists (since Δ^2 approaches $p^2/2$ at large p') and will be of the form of a function of g bounded by a polynomial in p times t times a constant that approaches unity exponentially fast. It should be noted that the six Gaussians in (3.83) of width \sqrt{t} each cancel the $1/t^3$ in the measure $d\Omega/t^3$ (recall from (3.18), (3.19) that $\|\psi_g^t\|^2 \nu_t(g)$ approaches $2/(\pi t^3)$ exponentially fast).

It follows that as far as the leading order (in t) behaviour of the expectation value of any monomial is concerned, we can drop the term $\Delta < \hat{p}_j >_{gg'}^t$ from $< \hat{p}_j >_{g,g'}^t$, at least in the range $0 \leq \phi \leq (1-c)\pi$.

Case B)

Let us now discuss the range $(1-c)\pi \leq \phi \leq \pi$. We will not be as explicit as in case A), the steps to be performed are essentially identical to the case A) and can be found in more detail in the analogous discussion of [41].

The essential point is now to write everything in terms of

$$z'_0 := z_0 - i\pi = s - i(\pi - \phi) =: s - i\phi' \text{ where } 0 \leq \phi' \leq c\pi \quad (3.85)$$

Starting with the expression (3.34) we have

$$\begin{aligned} < \hat{p}_j >_{gg'}^t = \left[\frac{-i \operatorname{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \right] \times \\ \times \frac{\left\{ \frac{t}{2} \sum_n e^{\frac{(z'_0 - i\pi(2n-1))^2 - p^2/2 - (p')^2/2}{t}} \left[2 \frac{(z'_0 - i\pi(2n-1))^2}{t \sinh(z_0)} - (z'_0 - i\pi(2n-1)) \frac{\cosh(z_0)}{\sinh^2(z_0)} + \frac{1}{\sinh(z_0)} \right] \right\}}{\left[\sum_n \frac{p-2\pi in}{\sinh(p)} e^{-\frac{4\pi^2 n^2}{t}} e^{-i\frac{4\pi n p}{t}} \right]^{1/2} \left[\sum_n \frac{p'-2\pi in}{\sinh(p')} e^{-\frac{4\pi^2 n^2}{t}} e^{-i\frac{4\pi n p'}{t}} \right]^{1/2}} \end{aligned} \quad (3.86)$$

$$\begin{aligned}
&= \left[\frac{-i \operatorname{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \right] \times \\
&\times e^{\frac{(z'_0)^2 - p^2/2 - (p')^2/2}{t}} \frac{\left\{ \frac{t}{2} \sum_{n=\text{odd}} e^{-\frac{\pi^2 n^2}{t}} e^{-\frac{2i\pi z'_0 n}{t}} \left[2 \frac{(z'_0 - i\pi n)^2}{t \sinh(z'_0)} - (z'_0 - i\pi n) \frac{\cosh(z_0)}{\sinh^2(z_0)} + \frac{1}{\sinh(z_0)} \right] \right\}}{\sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}} (1 - K_t(p))(1 - K_t(p'))}
\end{aligned}$$

where $|K_t(p)|, |K_t(p')| \leq K_t$ are the functions implicit in the estimate (3.35) bounded from above by exponentially vanishing constants. We now pull out a factor of $(z'_0)^2/\sinh(z_0)$ out of the series in (3.86), collect terms as to sum over positive, odd integers n only and observe that $\sinh(z_0) = -\sinh(z'_0)$ and $\cosh(z_0) = -\cosh(z'_0)$. Then (3.86) becomes

$$\begin{aligned}
\langle \hat{p}_j \rangle_{gg'}^t &= \frac{\left[\frac{-iz'_0 \operatorname{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \right] \left[\frac{z'_0}{\sinh(z'_0)} \right] e^{\frac{(z'_0)^2 - p^2/2 - (p')^2/2}{t}}}{\sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}} (1 - K_t(p))(1 - K_t(p'))} \times \\
&\times \sum_{n=1, \text{odd}}^{\infty} e^{-\frac{\pi^2 n^2}{t}} \left\{ 2 \left[\cos\left(\frac{2\pi n z'_0}{t}\right) - \frac{(2\pi n)^2 \sin\left(\frac{2\pi n z'_0}{t}\right)}{t} \right] + t \left[\frac{1}{(z'_0)^2} - \frac{\cosh(z'_0)}{z'_0 \sinh(z'_0)} \right] \cos\left(\frac{2\pi n z'_0}{t}\right) \right. \\
&\left. + \pi n t \left[\frac{\cosh(z'_0)}{(z'_0)^2 \sinh(z'_0)} \sin\left(\frac{2\pi n z'_0}{t}\right) - \frac{2\pi n}{(z'_0)^2 t} \cos\left(\frac{2\pi n z'_0}{t}\right) \right] \right\}
\end{aligned} \tag{3.87}$$

which can now be estimated essentially as in case A). The most important differences are the following : First, lemma 3.1 has now to be replaced by $|z'_0/\sinh(z'_0)| \leq k_{1-c}s/\sinh(s)$ which can be seen by following the proof given there step by step. Secondly, by the methods given in [41] the Gaussian is now estimated from above by

$$e^{-\frac{\Delta^2 + 2\delta^2 + \tilde{\theta}^2}{2t}} \tag{3.88}$$

where necessarily $\tilde{\theta} \geq \pi/2$ and is therefore exponentially small for all values of g' in the range $(1-c)\pi \leq \phi \leq \pi$ provided that we choose $c < 1/2$ as we do. Finally, choosing as in [41] $c = 1/32$ (see specifically formula (4.44) there) and using the same estimates of case A) we can display (3.87) as a function of g, g' bounded by a polynomial in p, p' times a Gaussian in Δ^2 multiplied by an overall constant which decays exponentially fast to zero as $t \rightarrow 0$. Thus, (3.87) is an Ω/t^3 integrable function with respect to g' and the result of the integration is a function of g bounded by a polynomial in p times a constant which decays exponentially fast as $t \rightarrow 0$.

We conclude that the range of integration $(1-c)\pi \leq \phi \leq \pi$ is irrelevant for the expectation value and all its corrections in powers of t .

We can now finish the estimate of the matrix element. Our discussion has demonstrated that to leading order in t we can replace $\langle \hat{p}_j \rangle_{gg'}^t$ by

$$\frac{\left[\frac{-i \operatorname{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} z_0 \right] e^{\frac{z_0^2 - p^2/2 - (p')^2/2}{t}} \frac{z_0}{\sinh(z_0)} \Theta((1-c)\pi - \phi)}{\sqrt{D^t(p) D^t(p')}} \tag{3.89}$$

where Θ is the step function. The expression (3.89) is Gaussian peaked at $g = g'$ with decay width of order \sqrt{t} at which it equals p_j . In order to perform the integral over g' we will therefore expand the square bracket in the numerator of (3.89) as

$$p_j + \left[\frac{-i \operatorname{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} z_0 - p_j \right] \tag{3.90}$$

and the integral over the g' independent term with respect to $\Omega(g')/t^3$ converges exponentially fast to p_j since the absolute value squared of (3.89) modulo the square bracket equals precisely the overlap function of [41] modulo a multiplicative term whose absolute value can be estimated from above by a constant that approaches unity exponentially fast. The integral over the remaining term can be expanded as a function of $\vec{p} - \vec{p}', h(h')^{-1}$ and vanishes at least linearly in t by standard properties of Gaussian integrals.

Collecting all the results we have arrived at the first main theorem of this paper.

Theorem 3.2 *The matrix elements of the momentum operators with respect to coherent states can be estimated by*

$$\left| \frac{\langle \psi_g^t, \hat{p}_j \psi_{g'}^t \rangle}{\|\psi_g^t\| \|\psi_{g'}^t\|} - p_j(g) \frac{\langle \psi_g^t, \psi_{g'}^t \rangle}{\|\psi_g^t\| \|\psi_{g'}^t\|} \right| \leq t f(\vec{p}, \vec{p}') \frac{|\langle \psi_g^t, \psi_{g'}^t \rangle|}{\|\psi_g^t\| \|\psi_{g'}^t\|} \quad (3.91)$$

where f is a polynomial of p, p' .

As a corollary to theorem (3.2) we obtain that the expectation value $\langle \hat{p}_j \rangle_{gg}^t$ equals $p_j(g)$ up to bounded corrections in $p_j(g)$ and that are proportional to t . We will actually calculate the exact correction in a later section by a different method.

3.1.3 Matrix Elements for the Holonomy Operator

The computation of the matrix element of the holonomy operator

$$\langle \hat{h}_{AB} \rangle_{gg'}^t := \frac{\langle \psi_g^t, \hat{h}_{AB} \psi_{g'}^t \rangle}{\|\psi_g^t\| \|\psi_{g'}^t\|} \quad (3.92)$$

turns out to be rather messy. Let us determine first the following matrix element, using the Peter&Weyl theorem and the $SL(2, \mathbb{C})$ identity $\chi_j(g) = \chi_j(g^{-1})$

$$\begin{aligned} & \langle \psi_g^t, \hat{h}_{A_0 B_0} \psi_{g'}^t \rangle \quad (3.93) \\ &= \sum_{j, j'} d_j d_{j'} e^{-t[\lambda_j + \lambda_{j'}]/2} \overline{\pi_j(g)_{A_1 \dots A_{2j}, B_1 \dots B_{2j}} \pi_{j'}(g')_{A'_1 \dots A'_{2j'}, B'_1 \dots B'_{2j'}}} \times \\ & \times \langle \pi_{j'}(\cdot)_{A'_1 \dots A'_{2j'}, B'_1 \dots B'_{2j'}}, \hat{h}_{A_0 B_0} \pi_j(\cdot)_{A_1 \dots A_{2j}, B_1 \dots B_{2j}} \rangle \\ &= \sum_{j, j'} d_j d_{j'} e^{-t[\lambda_j + \lambda_{j'}]/2} \overline{\pi_j(g)_{A_1 \dots A_{2j}, B_1 \dots B_{2j}} \pi_{j'}(g')_{A'_1 \dots A'_{2j'}, B'_1 \dots B'_{2j'}}} \times \\ & \times \left[\frac{\delta_{j', j+\frac{1}{2}}}{d_{j'}} \pi_{j'}(1)_{A'_1 \dots A'_{2j'}, A_0 A_1 \dots A_{2j}} \pi_{j'}(1)_{B'_1 \dots B'_{2j'}, B_0 B_1 \dots B_{2j}} + \frac{\delta_{j', j-\frac{1}{2}} d_{j-\frac{1}{2}}}{d_j d_{j'}} \times \right. \\ & \times \left. \pi_{j'}(1)_{A'_1 \dots A'_{2j'}, (A_2 \dots A_{2j} \in A_1) A_0} \pi_{j'}(1)_{B'_1 \dots B'_{2j'}, (B_2 \dots B_{2j} \in B_1) B_0} \right] \\ &= \sum_j d_j e^{-t\lambda_j/2} \overline{\pi_j(\bar{g}^T)_{B_1 \dots B_{2j}, A_1 \dots A_{2j}}} \times \\ & \times \left[e^{-t\lambda_{j+\frac{1}{2}}/2} \pi_{j+\frac{1}{2}}(g')_{A_0 A_1 \dots A_{2j}, B_0 B_1 \dots B_{2j}} - \frac{d_{j-\frac{1}{2}}}{d_j} e^{-t\lambda_{j-\frac{1}{2}}/2} \epsilon_{A_0(A_1 \pi_{j-\frac{1}{2}}(g')_{A_2 \dots A_{2j}}, (B'_2 \dots B'_{2j} \in B_1) B_0} \right] \end{aligned}$$

In the second step we have recalled the following (Clebsch-Gordan) identity, valid for arbitrary $g \in G^\mathbb{C} = SL(2, \mathbb{C})$ and proved in [41]

$$\begin{aligned} & g_{A_0 B_0} \pi_j(g)_{A_1 \dots A_{2j}, B_1 \dots B_{2j}} \\ &= \pi_{j+\frac{1}{2}}(g)_{A_0 \dots A_{2j}, B_0 \dots B_{2j}} - \frac{d_{j-\frac{1}{2}}}{d_j} \epsilon_{A_0(A_1 \pi_{j-\frac{1}{2}}(g)_{A_2 \dots A_{2j}}, (B_2 \dots B_{2j} \in B_1) B_0} \quad (3.94) \end{aligned}$$

with $A, B, .. = \pm 1/2$, round brackets around groups of indices denote total symmetrization taken as an idempotent operation, ϵ_{AB} is the skew symmetric spinor of rank two, $d_j = \dim(\pi_j) = 2j + 1$ and the relation with the usual matrix elements of the irreducible representation π_j is given by $\pi_j(g)_{A_1+...+A_{2j}, B_1+...+B_{2j}} = \pi_j(g)_{A_1..A_{2j}, B_1..B_{2j}}$.

The huge amount of summation indices that appear in (3.93) and which we cannot nicely contract as irreducible representations of different dimension are multiplied with each other make (3.93) impossible to work with because then we cannot apply the Weyl character – and Poisson summation formula, our main tools in all the estimates. Fortunately, we have the following trick at our disposal (it obviously extends to groups of higher rank) :

Let Δ_h be the Laplacian on $G = SU(2)$ acting on $h \in G$. Since $\pi_j(hg)_{mn} = \pi_j(h)_{mm'}\pi_j(g)_{m'n}$ is an eigenstate of $-\Delta_h$ with eigenvalue $j(j+1)$ we obtain the following formulas that isolate the irreducible pieces on the right hand side of (3.94)

$$\begin{aligned} & \{(j + \frac{1}{4})(hg)_{A_0B_0} + [-\Delta_h, (hg)_{A_0B_0}]\}\pi_j(hg)_{A_1..B_{2j}} \\ &= d_j\pi_{j+\frac{1}{2}}(hg)_{A_0..A_{2j}, B_0..B_{2j}} \end{aligned} \quad (3.95)$$

$$\begin{aligned} & \{(j + \frac{3}{4})(hg)_{A_0B_0} - [-\Delta_h, (hg)_{A_0B_0}]\}\pi_j(hg)_{A_1..B_{2j}} \\ &= -d_{j-\frac{1}{2}}\epsilon_{A_0(A_1}\pi_{j-\frac{1}{2}}(hg)_{A_2..A_{2j}), (B_2..B_{2j}}\epsilon_{B_1)B_0} \end{aligned} \quad (3.96)$$

Taking the limit $h \rightarrow 1$ in (3.95), (3.96) we can cast (3.93) into the following simpler form which allows us to contract indices

$$\begin{aligned} & < \psi_g^t, \hat{h}_{A_0B_0} \psi_{g'}^t > \\ &= \sum_j e^{-t\lambda_j/2} \pi_j(\bar{g}^T)_{B_1..B_{2j}, A_1..A_{2j}} \times \\ &\times \{ [e^{-t\lambda_{j+\frac{1}{2}}/2} ((j + \frac{1}{4})(hg')_{A_0B_0} + [-\Delta_h, (hg')_{A_0B_0}]) \\ &+ e^{-t\lambda_{j-\frac{1}{2}}/2} ((j + \frac{3}{4})(hg')_{A_0B_0} - [-\Delta_h, (hg')_{A_0B_0}])] \pi_j(hg')_{A_1..A_{2j}, B_1..B_{2j}} \}_{|h=1} \\ &= \sum_j e^{-t\lambda_j/2} \times \\ &\times \{ [e^{-t\lambda_{j+\frac{1}{2}}/2} ((j + \frac{1}{4})(hg')_{A_0B_0} + [-\Delta_h, (hg')_{A_0B_0}]) \\ &+ e^{-t\lambda_{j-\frac{1}{2}}/2} ((j + \frac{3}{4})(hg')_{A_0B_0} - [-\Delta_h, (hg')_{A_0B_0}])] \chi_j(hg' \bar{g}^T) \}_{|h=1} \\ &= (g')_{A_0B_0} \sum_j e^{-t\lambda_j/2} [e^{-t\lambda_{j+\frac{1}{2}}/2} (j + \frac{1}{4}) + e^{-t\lambda_{j-\frac{1}{2}}/2} (j + \frac{3}{4})] \chi_j(g' \bar{g}^T) \\ &+ \{ [-\Delta_h, (hg')_{A_0B_0}] \sum_j e^{-t\lambda_j/2} [e^{-t\lambda_{j+\frac{1}{2}}/2} - e^{-t\lambda_{j-\frac{1}{2}}/2}] \chi_j(hg' \bar{g}^T) \}_{|h=1} \end{aligned} \quad (3.97)$$

Formula (3.97) can be further simplified by making use of the commutator identity (use $\Delta_h = X_h^j X_h^j$)

$$\{ [-\Delta_h, (hg')_{A_0B_0}] f(h) \}_{|h=1} = \frac{3}{4} g'_{A_0B_0} f(1) - \frac{1}{2} (\tau_j g')_{A_0B_0} \left(\frac{d}{ds} \right)_{s=0} f(e^{s\tau_j}) \quad (3.98)$$

valid for any differentiable function of h . Inserting this into (3.97) results in the final formula

$$< \psi_g^t, \hat{h}_{AB} \psi_{g'}^t >$$

$$\begin{aligned}
&= g'_{AB} \sum_j e^{-t\lambda_j/2} [(j+1)e^{-t\lambda_{j+\frac{1}{2}}/2} + je^{-t\lambda_{j-\frac{1}{2}}/2}] \chi_j(g' \bar{g}^T) \\
&- \frac{1}{2} (\tau_j g')_{AB} \left(\frac{d}{ds} \right)_{s=0} \sum_j e^{-t\lambda_j/2} [e^{-t\lambda_{j+\frac{1}{2}}/2} - e^{-t\lambda_{j-\frac{1}{2}}/2}] \chi_j(e^{s\tau_j} g' \bar{g}^T) \quad (3.99)
\end{aligned}$$

to which we can now apply the Weyl character formula.

Let again $\cosh(z) = \text{tr}(e^{s\tau_j} g' \bar{g}^T)/2$ and $\cosh(z_0) = \text{tr}(g' \bar{g}^T)/2$, then

$$\begin{aligned}
&< \psi_g^t, \hat{h}_{AB} \psi_{g'}^t > \\
&= \frac{g'_{AB}}{2 \sinh(z_0)} \sum_j [(d_j + 1)e^{-\frac{t}{4}(d_j^2 + d_j - 1/2)} + (d_j - 1)e^{-\frac{t}{4}(d_j^2 - d_j - 1/2)}] \sinh(d_j z_0) \\
&- \frac{1}{2} (\tau_j g')_{AB} \left(\frac{d}{ds} \right)_{s=0} \sum_j [e^{-\frac{t}{4}(d_j^2 + d_j - 1/2)} - e^{-\frac{t}{4}(d_j^2 - d_j - 1/2)}] \frac{\sinh(d_j z)}{\sinh(z)} \\
&= \frac{g'_{AB}}{4 \sinh(z_0)} e^{t/8} \sum_{n=1}^{\infty} e^{-tn^2/4} [(n+1)e^{-nt/4} + (n-1)e^{nt/4}] [e^{nz_0} - e^{-nz_0}] \\
&- \frac{1}{2} (\tau_j g')_{AB} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/8}}{2 \sinh(z)} \sum_{n=1}^{\infty} e^{-tn^2/4} [e^{-nt/4} - e^{nt/4}] [e^{nz} - e^{-nz}] \\
&= \frac{g'_{AB}}{4 \sinh(z_0)} e^{t/8} \sum_{n=1}^{\infty} e^{-tn^2/4} \{ [ne^{n(z_0-t/4)} + (-n)e^{(-n)(z_0-t/4)}] + [ne^{n(z_0+t/4)} + (-n)e^{(-n)(z_0+t/4)}] \\
&\quad + [e^{n(z_0-t/4)} + e^{(-n)(z_0-t/4)}] - [e^{n(z_0+t/4)} + e^{(-n)(z_0+t/4)}] \} \\
&- \frac{1}{2} (\tau_j g')_{AB} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/8}}{2 \sinh(z)} \sum_{n=1}^{\infty} e^{-tn^2/4} \{ [e^{n(z-t/4)} + e^{(-n)(z-t/4)}] - [e^{n(z+t/4)} + e^{(-n)(z+t/4)}] \} \\
&= \frac{g'_{AB}}{4 \sinh(z_0)} e^{t/8} \sum_{n=-\infty}^{\infty} e^{-tn^2/4} [(n+1)e^{n(z_0-t/4)} + (n-1)e^{n(z_0+t/4)}] \\
&- \frac{1}{2} (\tau_j g')_{AB} \left(\frac{d}{ds} \right)_{s=0} \frac{e^{t/8}}{2 \sinh(z)} \sum_{n=-\infty}^{\infty} e^{-tn^2/4} [e^{n(z-t/4)} - e^{n(z+t/4)}] \quad (3.100)
\end{aligned}$$

where in the last step we have recognized that the terms in the curly brackets add up to zero at $n = 0$ and that the terms with $(-n)$ as argument can be taken care of by extending the series to negative values of n . Introducing

$$T := \sqrt{t}/2, \quad z_0^\pm = z_0/T \pm T, \quad z^\pm = z/T \pm T \quad (3.101)$$

and remembering (3.31) we may write (3.100) in the form (notice that s is to be replaced by $-s$ as compared to (3.24))

$$\begin{aligned}
&< \psi_g^t, \hat{h}_{AB} \psi_{g'}^t > \\
&= \frac{g'_{AB}}{4 \sinh(z_0) T} e^{t/8} \sum_{n=-\infty}^{\infty} e^{-(nT)^2} [(nT) + T] e^{(nT)z_0^-} + [(nT) - T] e^{(nT)z_0^+} \\
&- \frac{1}{2} (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \left(\frac{d}{dz} \right)_{z=z_0} \frac{e^{t/8}}{2 \sinh(z)} \sum_{n=-\infty}^{\infty} e^{-(nT)^2} [e^{(nT)z^-} - e^{(nT)z^+}] \quad (3.102)
\end{aligned}$$

An appeal to the Poisson summation formula now reveals that

$$\begin{aligned}
&< \psi_g^t, \hat{h}_{AB} \psi_{g'}^t > \\
&= g'_{AB} \frac{e^{t/8}}{4 \sinh(z_0) T} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \left[\left(\frac{z_0^- - \frac{2\pi in}{T}}{4\sqrt{\pi}} + \frac{T}{2\sqrt{\pi}} \right) e^{(z_0^- - \frac{2\pi in}{T})^2/4} + \left(\frac{z_0^+ - \frac{2\pi in}{T}}{4\sqrt{\pi}} - \frac{T}{2\sqrt{\pi}} \right) e^{(z_0^+ - \frac{2\pi in}{T})^2/4} \right] \quad (3.103)
\end{aligned}$$

$$\begin{aligned}
& - \frac{e^{t/8}}{8 \sinh(z_0)} (\tau_j g')_{AB} \text{tr}(\tau_j g' \bar{g}^T) \left(\frac{d}{dz} \right)_{z=z_0} \frac{1}{\sinh(z)} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \left[\frac{e^{(z^- - \frac{2\pi in}{T})^2/4}}{2\sqrt{\pi}} - \frac{e^{(z^+ - \frac{2\pi in}{T})^2/4}}{2\sqrt{\pi}} \right] \\
& = \frac{\sqrt{\pi} e^{t/8}}{8 \sinh(z_0) T^3} [g'_{AB}] \sum_{n=-\infty}^{\infty} \left[(z_0 + T^2 - 2\pi in) e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} + (z_0 - T^2 - 2\pi in) e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \\
& - \frac{\sqrt{\pi} e^{t/8}}{8 \sinh(z_0) T^3} [(\tau_j g')_{AB} \text{tr}(\tau_j g' \bar{g}^T)] \left(\frac{d}{dz} \right)_{z=z_0} \frac{T^2}{\sinh(z)} \sum_{n=-\infty}^{\infty} \left[e^{\frac{(z - T^2 - 2\pi in)^2}{t}} - e^{\frac{(z + T^2 - 2\pi in)^2}{t}} \right] \\
& = \frac{\sqrt{\pi} e^{t/8}}{8 \sinh(z_0) T^3} [g'_{AB}] \sum_{n=-\infty}^{\infty} \left[(z_0 + T^2 - 2\pi in) e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} + (z_0 - T^2 - 2\pi in) e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \\
& - \frac{\sqrt{\pi} e^{t/8}}{8 \sinh(z_0) T^3} [(\tau_j g')_{AB} \text{tr}(\tau_j g' \bar{g}^T)] T^2 \left\{ - \frac{\cosh(z_0)}{\sinh^2(z_0)} \sum_{n=-\infty}^{\infty} \left[e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} - e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \right. \\
& \quad \left. + \frac{1}{2T^2 \sinh(z_0)} \sum_{n=-\infty}^{\infty} \left[(z_0 - T^2 - 2\pi in) e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} - (z_0 + T^2 - 2\pi in) e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \right\}
\end{aligned}$$

Recalling the norm of the coherent states from (3.29),

$$\|\psi_g^t\|^2 = \frac{\sqrt{\pi} e^{t/4}}{4 \sinh(p) T^3} \sum_n (p - 2\pi in) e^{\frac{(p - 2\pi in)^2}{t}} =: \frac{\sqrt{\pi} e^{t/4}}{4 T^3} \frac{p}{\sinh(p)} e^{p^2/t} (1 + K_t(p)) \quad (3.104)$$

we arrive at the final exact formula for the matrix element of the holonomy operator (3.92) (all sums run over $n \in \mathbb{Z}$)

$$\begin{aligned}
& < \hat{h}_{AB} >_{gg'}^t \\
& = \frac{e^{-t/8} e^{-\frac{p^2 + (p')^2}{2t}}}{2 \sinh(z_0)} \times \\
& \times \left\{ g'_{AB} \sum_n \left[(z_0 + T^2 - 2\pi in) e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} + (z_0 - T^2 - 2\pi in) e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \right. \\
& - (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \sum_n \left[(z_0 - T^2 - 2\pi in - 2T^2 \frac{\cosh(z_0)}{\sinh(z_0)}) e^{\frac{(z_0 - T^2 - 2\pi in)^2}{t}} \right. \\
& \quad \left. \left. - (z_0 + T^2 - 2\pi in - 2T^2 \frac{\cosh(z_0)}{\sinh(z_0)}) e^{\frac{(z_0 + T^2 - 2\pi in)^2}{t}} \right] \right\} \quad (3.105)
\end{aligned}$$

At this point we must again distinguish between the cases A) $0 \leq \phi \leq (1 - c)\pi$ and B) $(1 - c)\pi \leq \phi \leq \pi$ for some $c < 1/2$ where $z_0 = s + i\phi$.

Case A)

We have

$$(z_0 \pm T^2 - 2\pi in)^2/t = \frac{z_0^2}{t} + \frac{t}{16} - \frac{4\pi^2 n^2}{t} - \frac{4\pi in z_0}{t} \pm (z_0/2 - i\pi n) \quad (3.106)$$

and thus can write (3.105) more explicitly as

$$\begin{aligned}
& < \hat{h}_{AB} >_{gg'}^t \\
& = \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2z_0^2}{2t}}}{2 \sinh(z_0)} \times \\
& \times \left\{ g'_{AB} \sum_n (-1)^n e^{-\frac{4\pi^2 n^2}{t}} e^{-\frac{4\pi in z_0}{t}} \left[(z_0 + T^2 - 2\pi in) e^{-z_0/2} + (z_0 - T^2 - 2\pi in) e^{z_0/2} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& - (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \sum_n (-1)^n e^{-\frac{4\pi^2 n^2}{t}} e^{-\frac{4\pi i n z_0}{t}} \left[(z_0 - T^2 - 2\pi i n - 2T^2 \frac{\cosh(z_0)}{\sinh(z_0)}) e^{-z_0/2} \right. \\
& \quad \left. - (z_0 + T^2 - 2\pi i n - 2T^2 \frac{\cosh(z_0)}{\sinh(z_0)}) e^{z_0/2} \right] \} \quad (3.107)
\end{aligned}$$

Let us focus on the curly bracket in (3.107) for which we find, after some considerable amount of algebra,

$$\begin{aligned}
& \{.\} = 2z_0 \times \\
& \times \left\{ g'_{AB} \left\{ \cosh(z_0/2) + \left[-\frac{T^2 \sinh(z_0/2)}{z_0/2} \right] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} \times \right. \right. \\
& \times \left. \left[\left(\cosh(z_0/2) - \frac{T^2 \sinh(z_0/2)}{z_0/2} \right) \cos\left(\frac{4\pi n z_0}{t}\right) \right] + \left[-\frac{8\pi^2 n^2}{t} \cosh(z_0/2) \frac{\sin\left(\frac{4\pi n z_0}{t}\right)}{\frac{4\pi n z_0}{t}} \right] \right\} \\
& - (\tau_j g')_{AB} z_0 \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \left\{ -\sinh(z_0/2)/z_0 + \left[2T^2 \frac{\coth(z_0) \sinh(z_0/2) - \frac{1}{2} \cosh(z_0/2)}{z_0^2} \right] \right. \\
& + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} \times \\
& \times \left[\left(-\sinh(z_0/2)/z_0 + 2T^2 \frac{\coth(z_0) \sinh(z_0/2) - \frac{1}{2} \cosh(z_0/2)}{z_0^2} \right) \cos\left(\frac{4\pi n z_0}{t}\right) \right] \\
& + \left[\frac{(2\pi n)^2 \sinh(z_0/2) \sin\left(\frac{4\pi n z_0}{t}\right)}{t \frac{4\pi n z_0}{t}} \right] \left. \right\} \\
& =: 2z_0 \{ g'_{AB} \{ \cosh(z_0/2) + [I_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([I_2] + [I_3]) \} \\
& - (\tau_j g')_{AB} z_0 \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \left\{ -\frac{\sinh(z_0/2)}{z_0} + 2T^2 [J_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([J_2] + [J_3]) \right\} \} \quad (3.108)
\end{aligned}$$

where we have abbreviated the terms in the square brackets in the first equality by $I_1, I_2, I_3, J_1, J_2, J_3$ in this order since we wish to estimate them separately.

Combining (3.107) with (3.108) yields

$$\begin{aligned}
& < \hat{h}_{AB} >_{gg'}^t \quad (3.109) \\
& = \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2z_0^2}{2t}} z_0}{\sinh(z_0)} \times \\
& \times \frac{\sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))}}{\sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))}} \times \\
& \times \left\{ g'_{AB} \left\{ \cosh(z_0/2) + [I_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([I_2] + [I_3]) \right\} \right. \\
& - (\tau_j g')_{AB} z_0 \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \left\{ -\sinh(z_0/2)/z_0 + [J_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([J_2] + [J_3]) \right\} \left. \right\}
\end{aligned}$$

and proceeding as in section 3.1.2 we define

$$\begin{aligned}
& \Delta < \hat{h}_{AB} >_{gg'}^t \\
& := < \hat{h}_{AB} >_{gg'}^t - \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2z_0^2}{2t}} z_0}{\sinh(z_0)} \times \\
& \times \left[g'_{AB} \left\{ \cosh(z_0/2) + (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \sinh(z_0/2) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2z_0^2}{2t}} z_0}{\sinh(z_0)} \times \\
&\quad \sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))} \times \\
&\quad \times \{g'_{AB} \{ [I_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([I_2] + [I_3]) \} \\
&\quad - (\tau_j g')_{AB} z_0 \frac{\text{tr}(\tau_j g' \bar{g}'^T)}{2 \sinh(z_0)} \{ 2T^2 [J_1] + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\frac{4\pi^2 n^2}{t}} ([J_2] + [J_3]) \} \} \quad (3.110)
\end{aligned}$$

The tools to estimate these terms have already been laid in section 3.1.2 so that we can be brief here. We just need the following result.

Lemma 3.6 *For any $z_0 = s + i\phi$, $0 \leq \phi \leq \pi(1 - c)$ we have*

$$|J_1| \leq \cosh(|z_0|/2) [\sqrt{2}k'_c + \frac{k_c k'_{\frac{1+c}{2}}}{4}] =: \tilde{k}_c \cosh(|z_0|/2) \quad (3.111)$$

Proof of Lemma 3.6 :

We easily establish the following identity

$$J_1 = \frac{1}{2} C(z_0) \frac{\sinh(z_0/2)}{z_0/2} - \frac{S(z_0)}{2} \cosh(z_0/2) + \frac{1}{8} \frac{z_0}{\sinh(z_0)} \frac{\sinh(z_0/2)}{z_0/2} [S(z_0/2) - C(z_0/2)] \quad (3.112)$$

Noticing that $\Im(z_0/2) \leq \pi(1 - c)/2 = \pi(1 - \frac{1+c}{2})$ the assertion follows from the lemmas of the previous subsection.

□

With the help of this lemma one finds

$$\begin{aligned}
|I_1| &= \left| -\frac{T^2 \sinh(z_0/2)}{2 z_0/2} \right| \leq \frac{T^2}{2} \cosh(|z_0|/2) \\
|I_2| &= \left| \left(\cosh(z_0/2) - \frac{T^2 \sinh(z_0/2)}{2 z_0/2} \right) \cos\left(\frac{4\pi n z_0}{t}\right) \right| \\
&\leq \cosh(|z_0|/2) (1 + T^2/2) \left| \cos\left(\frac{4\pi n z_0}{t}\right) \right| \leq \cosh(|z_0|/2) (1 + T^2) e^{\frac{4\pi^2 n(1-c)}{t}} \\
|I_3| &= \left| \frac{8\pi^2 n^2}{t} \cosh(z_0/2) \frac{\sin\left(\frac{4\pi n z_0}{t}\right)}{\frac{4\pi n z_0}{t}} \right| \\
&\leq \frac{8\pi^2 n^2}{t} \cosh(|z_0|/2) \left| \frac{\sin\left(\frac{4\pi n z_0}{t}\right)}{\frac{4\pi n z_0}{t}} \right| \leq \frac{16\pi^2 n^2}{t} \cosh(|z_0|/2) e^{\frac{4\pi^2 n(1-c)}{t}} \\
|J_2| &= \left| \left(-\sinh(z_0/2)/z_0 + 2T^2 J_1 \right) \cos\left(\frac{4\pi n z_0}{t}\right) \right| \\
&\leq \left| (2 \cosh(|z_0|/2) + 2T^2 |J_1|) \right| \left| \cos\left(\frac{4\pi n z_0}{t}\right) \right| \\
&\leq 2 \cosh(|z_0|/2) (1 + T^2 \tilde{k}_c) e^{\frac{4\pi^2 n(1-c)}{t}} \\
|J_3| &= \left| \frac{(2\pi n)^2 \sinh(z_0/2)}{t z_0/2} \frac{\sin\left(\frac{4\pi n z_0}{t}\right)}{\frac{4\pi n z_0}{t}} \right| \\
&\leq \frac{(2\pi n)^2}{t} \cosh(|z_0|/2) e^{\frac{4\pi^2 n(1-c)}{t}} \quad (3.113)
\end{aligned}$$

Observing the basic estimates (exploit $(\tau_j)^2 = -1$)

$$\begin{aligned}
|g'_{AB}|^2 &\leq \sum_{A,B} |g'_{AB}|^2 = \text{tr}(g'(\bar{g}')^T) = 2 \cosh(p') \leq 4 \cosh^2(p'/2) \text{ and} \\
|(\tau_j g')_{AB}|^2 &\leq \text{tr}(\tau_j g' (\overline{\tau_j g'})^T) = -\text{tr}(\tau_j^2 g'(\bar{g}')^T) \leq 4 \cosh^2(p'/2) \quad (3.114)
\end{aligned}$$

and employing the estimate for the denominator of (3.110) from the previous section together with (3.77) we obtain as an estimate (notation the same as in section 3.1.2)

$$\begin{aligned}
& |\Delta < \hat{h}_{AB} >_{gg'}^t| \tag{3.115} \\
& \leq \frac{e^{-t/16} e^{-\frac{p^2+(p')^2-2\Re(z_0^2)}{2t}} k_c \left| \frac{s}{\sinh(s)} \right| \cosh(|z_0|/2)}{(1-K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \{ |g'_{AB}| \{ \frac{T^2}{2} + 2 \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} e^{\frac{4\pi^2 n(1-c)}{t}} [1 + \frac{T^2}{2} + \frac{16\pi^2 n^2}{t}] \} \\
& + |(\tau_j g')_{AB}| |z_0| \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z_0)} \{ 2T^2 \tilde{k}_c + 2 \sum_{n=1}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} e^{\frac{4\pi^2 n(1-c)}{t}} [2(1 + T^2 \tilde{k}_c) + \frac{(2\pi n)^2}{t}] \} \} \\
& \leq \frac{2e^{-t/16} e^{-\frac{\Delta^2+2\delta^2+2\tilde{\theta}^2}{2t}} k_c \left| \frac{s}{\sinh(s)} \right| \cosh(|z_0|/2) \cosh(p'/2)}{(1-K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \{ \{ \frac{T^2}{2} + 2e^{-\frac{4\pi^2 c}{t}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} [1 + \frac{T^2}{2} + \frac{16\pi^2 (n+1)^2}{t}] \} \\
& + 4k_c \left| \frac{s}{\sinh(s)} \right| \cosh(\frac{p+p'}{2}) \{ 2T^2 \tilde{k}_c + 2e^{-\frac{4\pi^2 c}{t}} \sum_{n=0}^{\infty} e^{-\frac{4\pi^2 n^2}{t}} [2(1 + T^2 \tilde{k}_c) + \frac{(4\pi(n+1))^2}{t}] \} \} \}
\end{aligned}$$

In discussing the behaviour of this function as p' becomes large (integrability) we need to separate again the regions described by a) bounded δ (i.e. $s \approx \tilde{p}/2 \approx p'/2$) and b) unbounded δ (i.e. s/p' vanishes as $p' \rightarrow \infty$ or s is bounded) similar as in the previous section. In case a) (3.115) grows as $\cosh(|z_0|/2) \sqrt{p'}^3 \approx \cosh(p'/4) \sqrt{p'}^3$ times the Gaussian in $\Delta^2 + 2\tilde{\theta}^2$. In case b) it is damped by the Gaussian in δ^2 and is exponentially small times the Gaussian in $\Delta^2 + 2\tilde{\theta}^2$.

Finally looking at the terms inside the two inner curly brackets of (3.115) we see that they are up to a numerical factor given by constants of the form $t + K_t(c)$ where $K_t(c)$ vanishes exponentially fast as $t \rightarrow 0$. We conclude that the integral over g' of (3.115) in the range $0 \leq \phi \leq (1-c)\pi$ results in a function of g which is at most exponentially growing with p times a constant that approaches t exponentially fast.

Case B)

Following the by now already familiar trick we will now write (3.105) in terms of $z'_0 = z_0 - i\pi = s - i(\pi - \phi) = s - i\phi'$ with $0 \leq \phi' \leq c\pi$. This gives (observe that $\sinh(z_0) = -\sinh(z'_0)$, $\cosh(z_0) = -\cosh(z'_0)$)

$$\begin{aligned}
& < \hat{h}_{AB} >_{gg'}^t \\
& = - \frac{\frac{e^{-t/8} e^{-\frac{p^2+(p')^2}{2t}}}{2 \sinh(z'_0)}}{\sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))}} \times \\
& \times \{ g'_{AB} \sum_n [(z'_0 + T^2 - \pi i(2n-1)) e^{\frac{(z'_0 - T^2 - \pi i(2n-1))^2}{t}} + (z'_0 - T^2 - \pi i(2n-1)) e^{\frac{(z'_0 + T^2 - \pi i(2n-1))^2}{t}}] \\
& + (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \sum_n [(z'_0 - T^2 - \pi i(2n-1) - 2T^2 \frac{\cosh(z'_0)}{\sinh(z'_0)}) e^{\frac{(z'_0 - T^2 - \pi i(2n-1))^2}{t}} \\
& - (z'_0 + T^2 - \pi i(2n-1) - 2T^2 \frac{\cosh(z'_0)}{\sinh(z'_0)}) e^{\frac{(z'_0 + T^2 - \pi i(2n-1))^2}{t}}] \} \\
& = - \frac{\frac{e^{-t/8} e^{-\frac{p^2+(p')^2}{2t}}}{2 \sinh(z'_0)}}{\sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \{g'_{AB} \sum_{n=\text{odd}} [(z'_0 + T^2 - \pi in)e^{\frac{(z'_0 - T^2 - \pi in)^2}{t}} + (z'_0 - T^2 - \pi in)e^{\frac{(z'_0 + T^2 - \pi in)^2}{t}}] \\
& + (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \sum_{n=\text{odd}} [(z'_0 - T^2 - \pi in - 2T^2 \frac{\cosh(z'_0)}{\sinh(z'_0)})e^{\frac{(z'_0 - T^2 - \pi in)^2}{t}} \\
& - (z'_0 + T^2 - \pi in - 2T^2 \frac{\cosh(z'_0)}{\sinh(z'_0)})e^{\frac{(z'_0 + T^2 - \pi in)^2}{t}}]\} \quad (3.116)
\end{aligned}$$

With

$$\frac{(z'_0 \pm T^2 - \pi in)^2}{t} = \frac{(z'_0)^2}{t} + \frac{t}{16} - \frac{\pi^2 n^2}{t} - \frac{2\pi in z'_0}{t} \pm (z'_0/2 - i\pi n/2) \quad (3.117)$$

and since $i^{-n} = -i^n$ for n odd we find after some pages of algebra

$$\begin{aligned}
& < \hat{h}_{AB} >_{gg'}^t \\
& = - \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2(z'_0)^2}{2t}} z'_0}{2 \sinh(z'_0)} \times \\
& \quad \sqrt{\frac{p}{\sinh(p)} (1 + K_t(p)) \frac{p'}{\sinh(p')} (1 + K_t(p'))} \times \\
& \times \{ \{ 2g'_{AB} \sum_{n=1, \text{odd}} i^n e^{-\pi^2 n^2/t} ([2i \sinh(z'_0/2) \sin(2\pi n z'_0/t)] \\
& \quad + [-\pi n \cosh(z'_0/2) \frac{\sin(2\pi n z'_0/t)}{2\pi n z'_0/t}] + [4\pi in \frac{\sinh(z'_0/2)}{z'_0/2} \cos(2\pi n z'_0/t)]) \} \\
& + \{ (\tau_j g')_{AB} z'_0 \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \sum_{n=1, \text{odd}} i^n e^{-\pi^2 n^2/t} ([-\frac{4\pi in}{t} (2 \cosh(z'_0/2) \\
& \quad + T^2 \frac{\sinh(z'_0/2)}{z'_0/2}) \frac{\sin(2\pi n z'_0/t)}{2\pi n z'_0/t}] \\
& \quad + [2it \frac{\coth(z'_0) \sin(2\pi n z'_0/t) - \frac{2\pi n}{t} \cos(2\pi n z'_0/t)}{(z'_0)^2} \cosh(z'_0/2)]) \} \} \quad (3.118)
\end{aligned}$$

In estimating (3.118) the only term that is superficially non-regular at $z'_0 = 0$ is the last fraction in the second inner curly bracket. However, using the identity

$$\begin{aligned}
K & := \frac{\coth(z'_0) \sin(2\pi n z'_0/t) - \frac{2\pi n}{t} \cos(2\pi n z'_0/t)}{(z'_0)^2} = \frac{2\pi n}{t} \times \\
& \times \{ \{ [C(z'_0) \frac{\sin(\tilde{z}'_0)}{\tilde{z}'_0} - S(z'_0)] + (\frac{2\pi n}{t})^2 [s(\tilde{z}'_0) \frac{z'_0}{\sinh(z'_0)} - c(\tilde{z}'_0)] \} \} \quad (3.119)
\end{aligned}$$

where $\tilde{z}'_0 = 2\pi n z'_0/t$ and employing the estimates (3.66), (3.65) as well as lemmata 3.2, 3.4 it is not difficult to show that in the range $0 \leq \phi' \leq c\pi$

$$|K| \leq \frac{16\pi n}{t} [k'_{1-c} + (\frac{2\pi n}{t})^2 k_{1-c}] e^{2\pi^2 nc/t} \quad (3.120)$$

With these preparations and using previous results we can finish the estimate of (3.118)

$$\begin{aligned}
& | < \hat{h}_{AB} >_{gg'}^t | \quad (3.121) \\
& \leq \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2\Re((z'_0)^2)}{2t}} |z'_0|}{2 |\sinh(z'_0)|} \times \\
& \quad (1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}} \times
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ \{2|g'_{AB}| \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} ([2 \sinh(|z'_0|/2) |\sin(2\pi n z'_0/t)|] \right. \\
& \quad \left. + [\pi n \cosh(|z'_0|/2) \frac{\sin(2\pi n z'_0/t)}{2\pi n z'_0/t}] + [4\pi n \frac{\sinh(|z'_0|/2)}{|z'_0|/2} \cos(2\pi n z'_0/t)]]\} \right. \\
& + \left\{ |(\tau_j g')_{AB}| |z'_0| \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} \times \right. \\
& \quad \left. \times ([\frac{4\pi n}{t} (2 \cosh(|z'_0|/2) + T^2 \frac{\sinh(|z'_0|/2)}{|z'_0|/2})] \frac{\sin(2\pi n z'_0/t)}{2\pi n z'_0/t}] + [2t|K| \cosh(|z'_0|/2)]) \} \right\} \\
& \leq \cosh(|z'_0|/2) \frac{\frac{e^{-t/16} e^{-\frac{\Delta^2 + 2(\bar{p}^2/4 - s^2 + (\phi')^2)}{2t}} |z'_0|}{2 |\sinh(z'_0)|}}{(1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \left\{ \{2|g'_{AB}| \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} e^{2\pi^2 n c/t} (2 + \pi n + 4\pi n)\} \right. \\
& + \left\{ |(\tau_j g')_{AB}| |z'_0| \frac{\text{tr}(\tau_j g' \bar{g}^T)}{2 \sinh(z'_0)} \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} e^{2\pi^2 n c/t} (\frac{4\pi n}{t} [2 + T^2] + 32\pi n [k'_{1-c} + (\frac{2\pi n}{t})^2 k_{1-c}]) \} \right\} \\
& \leq \cosh(|z'_0|/2) k_{1-c} \frac{s}{\sinh(s)} \cosh(p'/2) \frac{e^{-t/16} e^{-\frac{\Delta^2 + 2(\bar{p}^2/4 - s^2 + (\phi')^2)}{2t}}}{(1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \left\{ \{2 \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} e^{2\pi^2 n c/t} (2 + 5\pi n)\} \right. \\
& + \left\{ 4k_{1-c} \frac{s}{\sinh(s)} \cosh(\frac{p+p'}{2}) \sum_{n=1, \text{odd}}^{\infty} e^{-\pi^2 n^2/t} e^{2\pi^2 n c/t} (\frac{4\pi n}{t} [2 + T^2] + 32\pi n [k'_{1-c} + (\frac{2\pi n}{t})^2 k_{1-c}]) \} \right\} \\
& \leq \cosh(|z'_0|/2) k_{1-c} \frac{s}{\sinh(s)} \cosh(p'/2) \frac{e^{-t/16} e^{-\frac{\Delta^2 + 2(\bar{p}^2/4 - s^2 + (\phi')^2 + (1-2c)\pi^2)}{2t}}}{(1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \left\{ \{2 \sum_{n=0, \text{even}}^{\infty} e^{-\pi^2 n^2/t} (2 + 5\pi(n+1))\} \right. \\
& + \left\{ 4k_{1-c} \frac{s}{\sinh(s)} \cosh(\frac{p+p'}{2}) \sum_{n=0, \text{even}}^{\infty} e^{-\pi^2 n^2/t} (\frac{4\pi(n+1)}{t} [2 + T^2] \right. \\
& \quad \left. + 32\pi(n+1) [k'_{1-c} + (\frac{2\pi(n+1)}{t})^2 k_{1-c}]) \} \right\}
\end{aligned}$$

where we have used in the last step that for any $n \geq 1$

$$-\pi^2 n^2/t + 2\pi^2 n c/t = -\pi^2 n(n-1)/t - \pi^2 n(1-2c)/t \leq -\pi^2 (n-1)^2/t - \pi^2 (1-2c)/t \quad (3.122)$$

using that $2c < 1$ and $(n-1)^2 \leq n(n-1)$ valid for $n \geq 1$. Consider now the piece of the argument of the Gaussian given by

$$\begin{aligned}
& -[\bar{p}^2/4 - s^2 + (\phi')^2 + (1-2c)\pi^2] = -[\bar{p}^2/4 - s^2 + \phi^2 - \tilde{\theta}^2] - \tilde{\theta}^2 + 2\pi\phi - 2(1-c)\pi^2 \\
& \leq -\delta^2 - \tilde{\theta}^2 + 2\pi\phi - 2(1-c)\pi^2 \leq -\delta^2 - \tilde{\theta}^2 + 2c\pi^2 = -\delta^2 - (1-d)\tilde{\theta}^2 - d\tilde{\theta}^2 + 2c\pi^2 \\
& \leq -\delta^2 - (1-d)\tilde{\theta}^2 - (d/4 - 2c)\pi^2 \quad (3.123)
\end{aligned}$$

where in the first inequality we used $\phi' = \pi - \phi$ and (3.81), in the second $\phi \leq \pi$ and in the third that $\phi < / = / > \pi/2$ iff $\tilde{\theta} < / = / > \pi/2$ so that for $2c < 1$ we have $\tilde{\theta} > \pi/2$. Here $0 < d < 1$ is an arbitrary real number. Choosing $c < d/8$, say $c = d/16$ and $d = 1/2$

for definiteness we can complete the estimate (3.127) by writing

$$\begin{aligned}
& | < \hat{h}_{AB} >_{gg'}^t | \\
& \leq \cosh(|z'_0|/2) k_{1-c} \frac{s}{\sinh(s)} \cosh(p'/2) \frac{e^{-t/16} e^{-\frac{\Delta^2 + 2\delta^2 + \tilde{\theta}^2}{2t}}}{(1 - K_t) \sqrt{\frac{p}{\sinh(p)} \frac{p'}{\sinh(p')}}} \times \\
& \times \left\{ \left\{ 2 \sum_{n=0, \text{even}}^{\infty} e^{-\pi^2 n^2 / t} (2 + 5\pi(n+1)) \right\} \right. \\
& + \left\{ 4k_{1-c} \frac{s}{\sinh(s)} \cosh\left(\frac{p+p'}{2}\right) \sum_{n=0, \text{even}}^{\infty} e^{-\pi^2 n^2 / t} \left(\frac{4\pi(n+1)}{t} [2 + T^2] \right. \right. \\
& \left. \left. + 32\pi(n+1) [k'_{1-c} + \left(\frac{2\pi(n+1)}{t}\right)^2 k_{1-c}] \right) \right\} \} \quad (3.124)
\end{aligned}$$

Comparing (3.115) with (3.124) we see that the overall structure is completely identical, the only essential difference being that $2\tilde{\theta}^2$ in the exponent of the Gaussian is replaced by $\tilde{\theta}^2$. Since now $\tilde{\theta} \geq \pi/2$ we conclude that the integral of (3.121) over g' with respect to Ω/t^3 exists, resulting in a function of g growing no stronger than exponentially with p times a constant that vanishes exponentially fast with $t \rightarrow 0$.

Summarizing, as far as the leading order behaviour (in t) of the matrix element of the holonomy operator is concerned, we can replace it by

$$\begin{aligned}
< \hat{h}_{AB} >_{gg'}^t & \approx \Theta(\pi(1-c) - \phi) \frac{e^{-t/16} e^{-\frac{p^2 + (p')^2 - 2z_0^2}{2t}} z_0}{\sinh(z_0)} \times \\
& \times \left[g'_{AB} \cosh(z_0/2) + (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}'^T)}{2 \sinh(z_0)} \sinh(z_0/2) \right] \quad (3.125)
\end{aligned}$$

The Gaussian displays a peak at $g = g'$ where $\tilde{p}/2 = p = p'$, $\tilde{\alpha} = \tilde{\theta} = 0$ implying $z_0 = p$ whence the prefactor in front of the square bracket in (3.125) becomes $e^{-t/16}$ and the square bracket itself becomes

$$\begin{aligned}
[.] & = g_{AB} \cosh(p/2) + (\tau_j g)_{AB} \frac{\text{tr}(\tau_j g \bar{g}^T)}{2 \sinh(p)} \sinh(p/2) \\
& = ([\cosh(p/2) + i \frac{p_j}{p} \sinh(p/2) \tau_j] g)_{AB} = (H(g)^{-1} g)_{AB} = h(g)_{AB} \quad (3.126)
\end{aligned}$$

as expected. We now write the square bracket as

$$h(g)_{AB} + [g'_{AB} \cosh(z_0/2) + (\tau_j g')_{AB} \frac{\text{tr}(\tau_j g' \bar{g}'^T)}{2 \sinh(z_0)} \sinh(z_0/2) - h(g)_{AB}]$$

and do the Gaussian in g' . Notice that the g' independent term is given by $h_{AB}(g) < 1 >_{gg'}^t$ while the integral over the additional term will be at least of order t since the first contribution to the Gaussian of width of order \sqrt{t} comes from quadratic terms in $\vec{p} - \vec{p}', \vec{\theta} - \vec{\theta}'$. This gives rise to the second main theorem.

Theorem 3.3 *The matrix elements of the holonomy operators with respect to coherent states can be estimated by*

$$\left| \frac{< \psi_g^t, \hat{h}_{AB} \psi_{g'}^t >}{||\psi_g^t|| ||\psi_{g'}^t||} - h_{AB}(g) \frac{< \psi_g^t, \psi_{g'}^t >}{||\psi_g^t|| ||\psi_{g'}^t||} \right| \leq t f'(\vec{p}, \vec{p}') \frac{< \psi_g^t, \psi_{g'}^t >}{||\psi_g^t|| ||\psi_{g'}^t||} \quad (3.127)$$

where f' is a function of \vec{p}, \vec{p}' growing no faster than exponentially in either or p, p' .

As a corollary to theorem (3.2) we obtain that the expectation value $\langle \hat{h}_{AB} \rangle_g^t$ equals $h_{AB}(g)$ up to bounded corrections to $h_{AB}(g)$ that are proportional to t . We will actually calculate the exact correction in the next section by a different method. Notice that due to the Gaussian behaviour of the overlap function the exponential growth of f' is irrelevant in computing the expectation value of operator monomials, the corrections of order at least t are always integrable.

3.1.4 Computation of Operator Monomial Expectation Values by a Different Method

One can compute the expectation value of operator monomials also by a different method which does not rely on the overcompleteness of the coherent states. To see how this works, notice first of all that due to $[\hat{p}_j, \hat{h}_{AB}] = it(\tau_j \hat{h})_{AB}/2$ every operator monomial can be reduced to finite linear combinations of operator monomials in the following “standard ordered form”

$$\langle (\hat{O}_1 \dots \hat{O}_{m+n})_0 \rangle_g^t = \langle \hat{p}_{j_1} \dots \hat{p}_{j_m} \hat{h}_{A_1 B_1} \dots \hat{h}_{A_n B_n} \rangle_g^t \quad (3.128)$$

for some $m, n \geq 0$ and some ordering of the j_1, \dots, j_m . The idea is now to use the following identity established in [41] for general semisimple compact G (we display the case $G = SU(2)$), relating the annihilation and holonomy operators,

$$\hat{g}_{AB} = e^{3t/8} (e^{-i\hat{p}_j \tau_j / 2} \hat{h})_{AB} \quad (3.129)$$

and to use the eigenvalue property of the coherent states $\hat{g}_{AB} \psi_g^t = g_{AB} \psi_g^t$. In order to do this we first have to invert (3.129) for \hat{h}_{AB} . The naive guess turns out to be the correct one.

Theorem 3.4 *The inversion of (3.129) reads*

$$\hat{h}_{AB} = e^{-3t/8} (e^{i\hat{p}_j \tau_j / 2} \hat{g})_{AB} \quad (3.130)$$

The proof of that theorem rests on the following lemma.

Lemma 3.7 *Define the strictly positive and self-adjoint operator \hat{p} by*

$$\hat{p} := \sqrt{\hat{p}_j \hat{p}_j + \frac{t^2}{4}} \quad (3.131)$$

which commutes with all the \hat{p}_j . Then the following operator identities hold for $G = SU(2)$ (obvious generalizations hold for groups of higher rank) :

$$\begin{aligned} e^{\pm i\hat{p}_j \tau_j / 2} &= e^{\pm \frac{t}{4}} \left\{ [\cosh(\hat{p}/2) \mp \frac{t \sinh(\hat{p}/2)}{2\hat{p}}] 1_2 \pm i\tau_j \hat{p}_j \frac{\sinh(\hat{p}/2)}{\hat{p}} \right\} \\ e^{i\hat{p}_j \tau_j / 2} e^{-i\hat{p}_j \tau_j / 2} &= e^{-i\hat{p}_j \tau_j / 2} e^{i\hat{p}_j \tau_j / 2} = 1_2 \hat{1}_{\mathcal{H}} \end{aligned} \quad (3.132)$$

Notice that no operator ordering ambiguities occur in (3.132).

Proof of Lemma 3.7 :

By definition (the operator \hat{p}_j is unbounded but the exponential can be defined by Nelson’s analytic vector theorem)

$$e^{\pm i\hat{p}_j \tau_j / 2} = \sum_{n=0}^{\infty} \frac{(\pm i/2)^n}{n!} (\hat{p}_j \tau_j)^n \quad (3.133)$$

Due to the Pauli matrix relation $\tau_j \tau_k = -\delta_{jk} 1_2 + \epsilon_{jkl} \tau_l$ and the commutator relation

$$\epsilon_{jkl} \hat{p}_j \hat{p}_k \tau_l = \frac{1}{2} \epsilon_{jkl} [\hat{p}_j, \hat{p}_k] \tau_l = -it \frac{1}{2} \epsilon_{jkl} \epsilon_{jkm} \hat{p}_m \tau_l = -it \hat{p}_j \tau_j \quad (3.134)$$

every power of the matrix valued operator $\hat{\underline{p}} = \tau_j \hat{p}_j$ can be written as a linear combination

$$(\hat{\underline{p}})^n = q_n(\hat{x})\hat{\underline{p}} + r_n(\hat{x})1_2\hat{1}_{\mathcal{H}} \text{ where } \hat{x} = -\hat{p}_j\hat{p}_j \quad (3.135)$$

and q_n, r_n are polynomials which are inductively defined by

$$(\hat{\underline{p}})^{n+1} = q_{n+1}(\hat{x})\hat{\underline{p}} + r_{n+1}(\hat{x})1_2\hat{1}_{\mathcal{H}} = [q_n(\hat{x})\hat{\underline{p}} + r_n(\hat{x})1_2\hat{1}_{\mathcal{H}}]\hat{\underline{p}} \quad (3.136)$$

from which we find

$$q_{n+1} = -q_n i t + r_n, \quad r_{n+1} = \hat{x} q_n, \quad q_1 = 1, \quad r_1 = 0 \quad (3.137)$$

Notice that no operator ordering problems arise.

As one can check, the two-dimensional recursion defined in (3.137) is solved by

$$q_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}, \quad r_n = -\lambda_+ \lambda_- \frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-} \quad (3.138)$$

where

$$\lambda_{\pm} = -i\left(\frac{t}{2} \pm \hat{p}\right) \quad (3.139)$$

Inserted back into (3.133) gives

$$e^{\pm i \hat{\underline{p}}/2} = \frac{1}{\lambda_+ - \lambda_-} [\hat{\underline{p}}(e^{\pm i \lambda_+/2} - e^{\pm i \lambda_-/2}) - 1_2 \hat{1}_{\mathcal{H}}(\lambda_- e^{\pm i \lambda_+/2} - \lambda_+ e^{\pm i \lambda_-/2})] \quad (3.140)$$

and writing out λ_{\pm} results in the first line of (3.132).

Using this result and that $[\hat{p}_j, \hat{p}] = 0$ we easily verify the second line in (3.132).

□

Remark :

In the form (3.132) the exponential of $\hat{\underline{p}}$ can directly be defined through the spectral theorem without recourse to Nelson's theorem.

Proof of Theorem 3.4 :

The proof follows trivially from the second line of (3.132).

□

Inserting formula (3.130) into (3.128) does not directly help us as not all the $\hat{g}_{A_k B_k}$ stand to the right. However, making use again of a finite number of commutation relations and the eigenvalue property we see that we have to leading order in t

$$\langle \hat{p}_{j_1} \cdots \hat{p}_{j_m} \hat{h}_{A_1 B_1} \cdots \hat{h}_{A_n B_n} \rangle_g^t = g_{C_1 B_1} \cdots g_{C_n B_n} \langle \hat{p}_{j_1} \cdots \hat{p}_{j_m} (e^{i \hat{\underline{p}}/2})_{A_1 C_1} \cdots (e^{i \hat{\underline{p}}/2})_{A_n C_n} \rangle_g^t [1 + O(t)] \quad (3.141)$$

Using the explicit expression (3.132) for $e^{i \hat{\underline{p}}/2}$ and $[\hat{p}_j, \hat{p}] = 0$ we see that we can compute the leading order of (3.141) once we know all expectation values of the form

$$\langle \hat{p}_{j_1} \cdots \hat{p}_{j_m} f(\hat{p}^2) \rangle_g^t = \langle f(\hat{p}^2) \hat{p}_{j_1} \cdots \hat{p}_{j_m} \rangle_g^t \quad (3.142)$$

where f is an arbitrary analytical function of \hat{p}^2 . If f would be at most a polynomial in \hat{p}^2 then it would suffice to know all the matrix elements of the form

$$\langle \hat{p}_{j_1} \cdots \hat{p}_{j_m} \rangle_g^t \quad (3.143)$$

however, the functions of \hat{p}^2 that appear in (3.132) are not simply polynomials. In what follows we will show that (3.142) can be computed once we know $\langle f(\hat{p}^2) \rangle_g^t$. The latter can then be computed by an appeal to the solution of the moment problem by Hamburger.

Recall that

$$\hat{p}_j \psi_g^t(h) = it/2(d/ds)_{s=0} \psi_g^t(e^{s\tau_j} h) = it/2(d/ds)_{s=0} \psi_{e^{-s\tau_j} g}^t(h) \quad (3.144)$$

and since \hat{p}_j is self-adjoint

$$\langle \hat{p}_{j_1} \dots \hat{p}_{j_m} f(\hat{p}^2) \rangle_g^t = \left(-\frac{it}{2}\right)^m \left[\left(\frac{d}{ds_m}\right)_{s_m=0} \dots \left[\left(\frac{d}{ds_1}\right)_{s_1=0} \frac{\langle \psi_{e^{-s_1\tau_{j_1}} \dots e^{-s_m\tau_{j_m}} g}^t, f(\hat{p}^2) \psi_g^t \rangle}{\|\psi_g^t\|^2} \right] \dots \right] \quad (3.145)$$

Let $g' = e^{-s_1\tau_{j_1}} \dots e^{-s_m\tau_{j_m}} g$, then since $\hat{p}\pi_{jmn}(h) = t(j+1/2)\pi_{jmn}(h)$ we have with $f(\cdot) = F \circ \sqrt{(\cdot)}$

$$\begin{aligned} \langle \psi_{g'}^t, F(\hat{p}) \psi_g^t \rangle &= \sum_j d_j e^{-t(d_j^2-1)/4} F(td_j/2) \chi_j(g' \bar{g}^T) \\ &= \frac{1}{2 \sinh(z)} \sum_j d_j e^{-t(d_j^2-1)/4} F(td_j/2) [e^{zd_j} - e^{-zd_j}] \\ &= \frac{e^{t/4}}{2 \sinh(z) T} \sum_{n \in \mathbb{Z}} e^{-(nT)^2/4} F((nT)T/2) e^{(nT)\frac{z}{T}} \end{aligned} \quad (3.146)$$

where $\cosh(z) = \text{tr}(g' \bar{g}^T)/2$ and $T = \sqrt{t}$. Applying the Poisson summation formula to (3.146) we find

$$\langle \psi_{g'}^t, F(\hat{p}) \psi_g^t \rangle = \frac{e^{t/4} 2 \sinh(zT^2)}{\sum_{n \in \mathbb{Z}}} g_n(z) \quad (3.147)$$

where

$$g_n(z) = \left[\int_{\mathbb{R}} dx e^{-ikx} g(x) \right]_{k=2\pi n/T} = \int_{\mathbb{R}} dx e^{-x^2/4} F(xT/2) x e^{x \frac{z-2\pi in}{T}} \quad (3.148)$$

Let $\cosh(z_0) = \text{tr}(g \bar{g}^T)/2$ and define $z = z(s_1, \dots, s_m)$, $z_k = z(s_1, \dots, s_k, 0, \dots, 0)$, $k = 0, \dots, m$. Let $G(z)$ be any function of z , then

$$\begin{aligned} &\left[\left(\frac{d}{ds_m}\right)_{s_m=0} \dots \left[\left(\frac{d}{ds_1}\right)_{s_1=0} G(z) \right] \dots \right] = \left[\left(\frac{d}{ds_m}\right)_{s_m=0} \dots \left[\left(\frac{d}{ds_2}\right)_{s_2=0} \left(\frac{dz}{ds_1}\right)_{s_1=0} G^{(1)}(z_{m-1}) \right] \dots \right] \\ &= \left[\left(\frac{d}{ds_m}\right)_{s_m=0} \dots \left[\left(\frac{d}{ds_3}\right)_{s_3=0} \left\{ \left(\frac{d^2 z}{ds_1 ds_2}\right)_{s_1=s_2=0} G^{(1)}(z_{m-2}) \right. \right. \right. \\ &\quad \left. \left. \left. + \left(\frac{dz}{ds_1}\right)_{s_1=s_2=0} \left(\frac{dz}{ds_2}\right)_{s_1=s_2=0} G^{(2)}(z_{m-2}) \right\} \right] \dots \right] \\ &= \dots \\ &= \left(\frac{d^m z}{ds_1 \dots ds_m}\right)_{s_1=\dots=s_m=0} G^{(1)}(z_0) + \dots + \left(\frac{dz}{ds_1}\right)_{s_1=\dots=s_m=0} \dots \left(\frac{dz}{ds_m}\right)_{s_1=\dots=s_m=0} G^{(m)}(z_0) \end{aligned} \quad (3.149)$$

Applied to $G(z) = \langle \psi_{g'}^t, F(\hat{p}) \psi_g^t \rangle$ we infer that the derivatives of $g_n(z)/\sinh(z)$ at z_0 of all orders k between 1 and m appear in (3.145) with coefficients that involve sums, over all partitions of $m = l_1 + \dots + l_k$, $l_j \geq 1$, of products of k factors of the form

$$\left(\frac{d^l z}{ds_{I_1} \dots ds_{I_l}}\right)_{s_1=\dots=s_m=0}, \quad 1 \leq I_1, \dots, I_l \leq m \text{ mutually disjoint} \quad (3.150)$$

Performing the k -th derivative of $g_n(z)$ we obtain

$$g_n^{(k)}(z_0) = \frac{1}{T^k} \int_{\mathbb{R}} dx e^{-x^2/4} F(xT/2) x^{k+1} e^{x \frac{z_0-2\pi in}{T}} \quad (3.151)$$

The idea is now to do integrations by parts until only x appears instead of x^{k+1} using $xe^{-x^2/4} = -2(e^{-x^2/4})'$. There are no boundary terms due to the Gaussian. Each time the derivative hits $e^{x\frac{z-2\pi in}{T}}$ it brings down a factor of $\frac{z-2\pi in}{T}$ while hitting $x^l F(xT/2)$ produces a non-negative power of T . We conclude that

$$\begin{aligned} g_n^{(k)}(z_0) &= \frac{(2[z_0 - 2\pi in])^k}{T^{2k}} \left[\int_{\mathbb{R}} dx e^{-x^2/4} F(xT/2) x e^{x\frac{z_0-2\pi in}{T}} \right] (1 + O(T)) \\ &= \frac{(2[z_0 - 2\pi in])^k}{T^{2k}} g_n(z_0) (1 + O(T)) \end{aligned} \quad (3.152)$$

Since in (3.145) we multiply with T^{2m} and since the derivatives of z at $s_1 = \dots = s_m = 0$ are independent of t we see that to leading order in t we only need to keep the term with $k = m$. The same argument reveals that we do not need to take into account the derivatives of $1/\sinh(z)$.

Next, by a substitution and a contour argument we obtain (at least for functions F integrable against the Gaussian)

$$g_n(z_0) = e^{4\frac{(z_0-2\pi in)^2}{t}} \int_{\mathbb{R}} dx e^{-x^2/4} F\left(T\left(x + \frac{z_0 - 2\pi in}{T}\right)/2\right) \left(x + \frac{z_0 - 2\pi in}{T}\right) \quad (3.153)$$

By our assumption on F the integral exists and by the already familiar argument, the $e^{z_0^2}$ in (3.153) is controlled by the $e^{-p^2/t}$ coming from the denominator $\|\psi_g^t\|^2$ in (3.142) so that (3.153) is exponentially suppressed with $t \rightarrow 0$ for any $n \neq 0$.

Putting everything together we therefore have to leading order in T

$$\begin{aligned} \langle \hat{p}_{j_1} \dots \hat{p}_{j_m} f(\hat{p}^2) \rangle_g^t &= (-it/2)^m (2z_0/t)^m \left[\prod_{l=1}^m \left(\frac{dz}{ds_l} \right)_{s_1=\dots=s_m=0} \right] \frac{\frac{e^{t/4}}{2\sinh(z_0)T^2} \sum_{n \in \mathbb{Z}} g_n(z_0)}{\|\psi_g^t\|^2} (1 + O(T)) \\ &= \left[\prod_{l=1}^m (-iz_0 \frac{dz}{ds_l})_{s_1=\dots=s_m=0} \right] \langle f(\hat{p}^2) \rangle_g^t (1 + O(T)) \end{aligned} \quad (3.154)$$

Now by the method of section 3.1.2 we find

$$z(s_1, \dots, s_m) = p + i \frac{1}{p} \sum_{l=1}^m s_l p_{j_l} + O(s^2) \quad (3.155)$$

and we obtain the desired result

$$\langle \hat{p}_{j_1} \dots \hat{p}_{j_m} f(\hat{p}^2) \rangle_g^t = p_{j_1}(g) \dots p_{j_m}(g) \langle f(\hat{p}^2) \rangle_g^t (1 + O(T)) \quad (3.156)$$

It therefore remains to compute the expectation value $\langle f(\hat{p}^2) \rangle_g^t$ where f for the purpose of computing (3.128) can be chosen analytical in $\hat{O} := \hat{p}^2$ and at most exponentially growing with \hat{p} . Consider first the expectation values of the powers \hat{O}^n . Since they are of the form (3.155) with the choice $f = 1$ we immediately find

$$\lim_{t \rightarrow 0} \langle \hat{O}^n \rangle_g^t = O(g)^n = (p_j(g) p_j(g))^n \quad (3.157)$$

The assertion $\lim_{t \rightarrow 0} \langle f(\hat{p}^2) \rangle_g^t = f(p_j(g) p_j(g))$ follows therefore immediately from the solution of the moment problem due to Hamburger which is the subject of the next sub-section.

Remark :

Obviously, since we can compute commutators of polynomial operators and express it in terms of elementary operators again, the Ehrenfest theorem to first order in t for such operators is trivially satisfied because the operator algebra of elementary operators precisely mirrors the classical Poisson algebra.

3.2 Expectation Values of Non-Polynomial Operators and the Moment Problem due to Hamburger

Recall the following theorem (see, e.g. [63])

Theorem 3.5 (Hamburger) *Let be given a sequence of real numbers $a_n \in \mathbb{R}$, $n = 0, 1, 2, \dots$. A necessary and sufficient criterion for the existence of a positive, finite measure $d\rho(x)$ on \mathbb{R} such that the a_n are its moments, that is,*

$$a_n = \int_{\mathbb{R}} d\rho(x) x^n \quad (3.158)$$

is that for any natural number $0 \leq N < \infty$ and arbitrary complex numbers z_k , $k = 0, \dots, N$ it holds that

$$\sum_{k,l=0}^N \bar{z}_k z_l a_{n+m} \geq 0 \quad (3.159)$$

The measure is faithful if equality in (3.159) occurs only for $z_k = 0$. Moreover, if there exist constants $\alpha, \beta > 0$ such that $|a_n| \leq \alpha \beta^n (n!)$ for all n , then the measure ρ is unique.

Necessity is easy to see by considering the L_2 norm of the functions $\sum_{k=0}^N z_k x^k$. Sufficiency follows from the spectral theorem and uniqueness can be established by an appeal to Nelson's analytic vector theorem.

In this section we assume that all operators under consideration are densely defined on a common domain which they together with arbitrary powers leave invariant. We are then able to prove the following theorem.

Theorem 3.6 *Let \hat{O} be a self-adjoint operator \hat{O} on \mathcal{H}_γ for some $\gamma \in \Gamma_0^\omega$ built from \hat{p}_j^e, \hat{h}_e , $e \in E(\gamma)$, that is, $\hat{O} = O(\{\hat{p}_e, \hat{h}_e\}_{e \in E(\gamma)})$. Let $O(\vec{g}) = O(\{p_e(g_e), h_e(g_e)\}_{e \in E(\gamma)})$ be its real valued classical counterpart and suppose that for every $n \in \mathbb{N}$*

$$\lim_{t \rightarrow 0} \langle \hat{O}^n \rangle_{\gamma, \vec{g}}^t = O(\vec{g})^n \quad (3.160)$$

Then for any Borel measurable function f on \mathbb{R} such that $\langle f(\hat{O})^\dagger f(\hat{O}) \rangle_g^t < \infty$ we have

$$\lim_{t \rightarrow 0} \langle f(\hat{O}) \rangle_{\gamma, \vec{g}}^t = f(O(\vec{g})) \quad (3.161)$$

Proof of Theorem 3.6 :

Let $E(x)$, $x \in \mathbb{R}$ be the spectral projections of \hat{O} . Then, by assumption and the spectral theorem

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} d \langle \xi_{\gamma, \vec{g}}^t, E(x) \xi_{\gamma, \vec{g}}^t \rangle x^n = O(\vec{g})^n \quad (3.162)$$

where $\xi_{\vec{g}}^t = \psi_{\vec{g}}^t / \|\psi_{\vec{g}}^t\|$. Now $a_n := O(\vec{g})^n$ obviously satisfies all the criteria of theorem 3.5 and we conclude that there exists a measure $d\rho_{\vec{g}}(x)$ on \mathbb{R} such that

$$\int_{\mathbb{R}} d\rho_{\vec{g}}(x) x^n = O(\vec{g})^n \quad (3.163)$$

The Dirac measure $d\rho_{\vec{g}}(x) = \delta_{\mathbb{R}}(x, O(\vec{g})) dx$ obviously satisfies (3.163) and choosing $\alpha = 1, \beta = |O(\vec{g})|$ in theorem 3.5 obviously satisfies the uniqueness part of the criterion. Thus, the Dirac measure is in fact the unique solution to our moment problem. Thus the spectral measure $d\rho_{\vec{g}}^t(x) := d \langle \xi_{\gamma, \vec{g}}^t, E(x) \xi_{\gamma, \vec{g}}^t \rangle$ approaches the Dirac δ distribution when evaluated on monomials x^n . It follows that the support of $\rho_{\vec{g}}^t$ gets confined to $\{x_0\}$ as $t \rightarrow 0$ by definition of the Lebesgue integral.

Now for any function f satisfying the assumptions of the theorem the spectral theorem applies and we have

$$\langle f(\hat{O}) \rangle_{\vec{g}}^t = \int_{\mathbb{R}} d\rho_{\vec{g}}^t(x) f(x) \quad (3.164)$$

Thus, since f is in particular measurable, the limit of both sides of (3.164) turns into (3.161).

□

Corollary 3.1 *Let $\hat{O}_1, \dots, \hat{O}_m$ be self-adjoint, not necessarily commuting, operators such that*

$$\lim_{t \rightarrow 0} \langle \prod_{k=1}^m \hat{O}_k^{n_k} \rangle_{\gamma, \vec{g}}^t = \prod_{k=1}^m O_k(\vec{g})^{n_k} \quad (3.165)$$

Then for any Borel measurable function f on \mathbb{R}^m

$$\lim_{t \rightarrow 0} \langle f(\{\hat{O}_k\}_{k=1}^m) \rangle_{\gamma, \vec{g}}^t = f(\{O_k(\vec{g})\}_{k=1}^m) \quad (3.166)$$

Proof of Corollary 3.1 :

By the spectral theorem

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^m} d^m x \langle E_1(x_1) \dots E_m(x_m) \rangle_{\vec{g}}^t = \prod_{k=1}^m \int_{\mathbb{R}} dx_k x_k^{n_k} E_k(x_k) = \prod_{k=1}^m O_k(\vec{g})^{n_k} \quad (3.167)$$

where $E_k(x)$ is the family of spectral projections of \hat{O}_k . Thus, by the unique solution to the moment problem the measure in (3.167) approaches the product Dirac measure $d^m x \prod_k \delta_{\mathbb{R}}(x_k, O_k(\vec{g}))$ similar as in theorem 3.6.

□

Next we turn to commutators.

Theorem 3.7 *Suppose that \hat{O}_1, \hat{O}_2 are self-adjoint operators satisfying the assumptions of corollary 3.1. Suppose, moreover, that \hat{O}_1 is positive semi-definite and that*

$$\lim_{t \rightarrow 0} \frac{\langle [\hat{O}_1, \hat{O}_2] \rangle_{\vec{g}}^t}{it} = \{O_1, O_2\}(\vec{g}) \quad (3.168)$$

Then for any real number r

$$\lim_{t \rightarrow 0} \frac{\langle [(\hat{O}_1)^r, \hat{O}_2] \rangle_{\vec{g}}^t}{it} = \{(O_1)^r, O_2\}(\vec{g}) \quad (3.169)$$

Proof of Theorem 3.7 :

It suffices to prove the theorem for rational $r = m/n$ with m, n integers and $n > 0$. We have the identity

$$\begin{aligned} \frac{[\hat{O}_1^m, \hat{O}_2]}{it} &= \sum_{k=1}^m \hat{O}_1^{k-1} \frac{[\hat{O}_1, \hat{O}_2]}{it} \hat{O}_1^{m-k} \\ &= \sum_{k=1}^n \hat{O}_1^{m(k-1)/n} \frac{[\hat{O}_1^r, \hat{O}_2]}{it} \hat{O}_1^{m(n-k)/n} \end{aligned} \quad (3.170)$$

Now for any measurable function f we have by assumption and completeness relation

$$\begin{aligned} \lim_{t \rightarrow 0} \langle f(\hat{O}_1) f(\hat{O}_1) \rangle_{\vec{g}}^t &= \lim_{t \rightarrow 0} \left(\frac{2}{\pi t^3} \right)^N \int_{(G^{\mathbb{C}})^N} d^N \Omega(\vec{g}') \langle f(\hat{O}_1) \rangle_{\vec{g} \vec{g}'}^t \langle f(\hat{O}_1) \rangle_{\vec{g}' \vec{g}}^t \\ &= \lim_{t \rightarrow 0} \langle f(\hat{O}_1) \rangle_{\vec{g} \vec{g}}^t \langle f(\hat{O}_1) \rangle_{\vec{g} \vec{g}}^t \end{aligned} \quad (3.171)$$

meaning that $(\frac{2}{\pi t^3})^N | \langle f(\hat{O}_1) \rangle_{\vec{g}}^t |^2$ approaches a delta distribution times $(\langle f(\hat{O}_1) \rangle_{\vec{g}}^t)^2$, for any f , with respect to Ω^N as $t \rightarrow 0$ where $N = |E(\gamma)|$. It follows that $\langle f(\hat{O}_1) \rangle_{\vec{g}}^t$ is concentrated at $g = g'$ as explicitly displayed in sections 3.1.2, 3.1.3 and we therefore find for the expectation value of (3.170) by using again the completeness relation in a similar fashion

$$m \lim_{t \rightarrow 0} \langle \hat{O}_1^{m-1} \rangle_{\vec{g}}^t \langle \frac{[\hat{O}_1, \hat{O}_2]}{it} \rangle_{\vec{g}}^t = n \lim_{t \rightarrow 0} \langle \hat{O}_1^{m(n-1)/n} \rangle_{\vec{g}}^t \langle \frac{[\hat{O}_1^r, \hat{O}_2]}{it} \rangle_{\vec{g}}^t \quad (3.172)$$

Using the assumptions of the theorem we thus find

$$\lim_{t \rightarrow 0} \langle \frac{[\hat{O}_1^r, \hat{O}_2]}{it} \rangle_{\vec{g}}^t = \frac{m}{n} O_1(\vec{g})^{\frac{m}{n}-1} \{O_1, O_2\}(\vec{g}) = \{O_1^r, O_2\}(\vec{g}) \quad (3.173)$$

as claimed.

□

The application of these theorems concerns operators which are not polynomials of the elementary ones. Such operators occur in quantum general relativity where diffeomorphism invariance requires that Hamiltonian constraint operators are density one valued and therefore free of UV singularities. This enforces that non-analytic functions, specifically roots of the volume operator [16, 18, 19, 20], appear. The spectral measure of this operator is not explicitly known and therefore a direct computation of its expectation values and its commutators with holonomy operators that appear in [21, 22, 23, 24, 26] is a hopeless task. Theorems 3.6, 3.7 and corollary 3.1 circumvent this problem at least as far as the leading order behaviour of expectation values is concerned by using the following trick : The fourth power of the volume operator $\hat{O} := \hat{V}^4$ is in fact a polynomial of the p_j^e and thus the expectation values of \hat{O}^n and the commutators with holonomy operators can be straightforwardly computed, leading to the expected result. Defining then $\hat{V} := \hat{O}^{1/4}$ and using the above results shows that the Hamiltonian constraint indeed has the correct classical limit. Details will appear elsewhere [51].

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A The $U(1)$ case

In this appendix we will apply the results of this paper to the case of $U(1)$ as the gauge group. As will become clear, the much simpler structure of $U(1)$ leads to a considerable simplification of the derivation of all the results. The main reason for this is, of course, the fact that $U(1)$ is Abelian and as a consequence of this that all its irreducible representations are one-dimensional. This means that one has to deal with numbers only, instead of matrices.

A.1 Expectation Values of the Momentum Operator

We will first show that the expectation value of the momentum operator with respect to the $U(1)$ coherent states has the proper (semi-)classical limit. Recall from [41] the form

of the coherent states for $U(1)$:

$$\psi_g^t(h) = \sum_n e^{-\frac{t}{2}n^2} (gh^{-1})^n \quad (\text{A.1})$$

with $g = e^p e^{i\theta_0}$ and $h = e^{i\theta}$. By the same token as in section 3.1.4, the expectation value of \hat{p} with respect to these states is given by :

$$\frac{\langle \psi_g^t, \hat{p} \psi_g^t \rangle}{\|\psi_g^t\|^2} = it \left(\frac{d}{dr} \right)_{r=0} \frac{\sum_n e^{-tn^2} (e^{-ir} e^{2p})^n}{\sum_n e^{-tn^2} e^{2np}} = \frac{\sum_n e^{-tn^2} nt e^{2np}}{\sum_n e^{-tn^2} e^{2np}} \quad (\text{A.2})$$

To see the behaviour of this expression for $t \rightarrow 0$, we have to perform a Poisson transformation. For the denominator this has already been done in the appendix of [41], the result being

$$\sum_n e^{-tn^2} e^{2np} = \sqrt{\pi} e^{\frac{p^2}{t}} \sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}. \quad (\text{A.3})$$

The transformation for the numerator, however, has to be calculated anew. With $s = \sqrt{t}$, as usual, we get

$$\begin{aligned} \tilde{f}(k) &= s \int dx x e^{-x^2} e^{(\frac{2p}{s} - ik)x} \\ &= s e^{\frac{(\frac{2p}{s} - ik)^2}{4}} \int dx \left(x - \frac{ik - 2p/s}{2} \right) e^{-x^2} \\ &= \sqrt{\pi} s \frac{(2p/s - ik)}{2} e^{\frac{(\frac{2p}{s} - ik)^2}{4}}. \end{aligned} \quad (\text{A.4})$$

Inserting this into (A.2) yields

$$\begin{aligned} \frac{\langle \psi_g^t, \hat{p} \psi_g^t \rangle}{\|\psi_g^t\|^2} &= \frac{s \sum_n (p - \pi ni) / s e^{\frac{(2p - 2\pi ni)^2}{4t}}}{e^{p^2/t} \sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}} \\ &= \frac{p \sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}} - 2\pi i \sum_n n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}}{\sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}} \\ &= p - \frac{2\pi i \sum_n n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}}{\sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{t}}}. \end{aligned} \quad (\text{A.5})$$

and this result makes it immediately obvious that

$$\lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{p} \psi_g^t \rangle}{\|\psi_g^t\|^2} = p \quad (\text{A.6})$$

Our next task is to generalize this calculation to an arbitrary integer power of the momentum operator. We thus have

$$\frac{\langle \psi_g^t, \hat{p}^m \psi_g^t \rangle}{\|\psi_g^t\|^2} = \frac{\sum_n e^{-tn^2} (nt)^m e^{2np}}{\sum_n e^{-tn^2} e^{2np}} \quad (\text{A.7})$$

which now has to be Poisson transformed again. The transform for the denominator stays the same, so we can concentrate on the numerator. As the general steps are the same as above we will be more concise here. The Poisson transform is given by

$$\begin{aligned} \tilde{f}(k) &= s^m e^{\frac{(ik - 2p/s)^2}{4}} \int dy e^{-y^2} (y + (p/s - ik/2))^m \\ &= s^m e^{\frac{(ik - 2p/s)^2}{4}} \int dy e^{-y^2} \sum_{l=0}^m y^{m-l} (p/s - ik/2)^l \binom{m}{l} \end{aligned} \quad (\text{A.8})$$

As we are only interested in the limit $t \rightarrow 0$ and due to the prefactor s^m the only surviving term will be the $l = m$ term. We therefore have

$$\begin{aligned} \lim_{t \rightarrow 0} \tilde{f}(k) &= s^m e^{\frac{(ik-2p/s)^2}{4}} \int dy e^{-y^2} (p/s - ik/2)^m \\ &= \sqrt{\pi} (p - i\pi n)^m e^{\frac{(2i\pi n - 2p)^2}{4t}} \end{aligned} \quad (\text{A.9})$$

where we already substituted the k variable by $2\pi n/\sqrt{t}$. This expression now has to be put back into (A.7) which yields

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{p}^m \psi_g^t \rangle}{\|\psi_g^t\|^2} &= \lim_{t \rightarrow 0} \frac{\frac{2\pi}{\sqrt{t}} \sqrt{\pi} e^{p^2/t} \sum_n (p - i\pi n)^m e^{-\frac{\pi^2 n^2 + 2\pi i n p}{t}}}{\frac{2\pi}{\sqrt{t}} \sqrt{\pi} e^{p^2/t} \sum_n e^{-\frac{\pi^2 n^2 + 2\pi i n p}{t}}} \\ &= p^m, \end{aligned} \quad (\text{A.10})$$

as the only term in the sum, which survives in the limit, is the one with $n = 0$.

Although these results are quite satisfying, one often encounters other powers of the momentum operators, especially square and higher roots, so it would be reassuring to know that they, too, have the expected semiclassical behaviour. A direct calculation as performed above becomes quite difficult for roots of arbitrary polynomials of \hat{p}_e where e labels the edges of a graph (for one edge and, say, $\sqrt{|\hat{p}|}$ the computational effort is still low and is left to the reader as an exercise) so we have to resort to other methods. A clue comes from reformulating the expectation value for integer powers of \hat{p} :

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{p}^m \psi_g^t \rangle}{\|\psi_g^t\|^2} &= \lim_{t \rightarrow 0} \frac{\sum_n \langle \psi_g^t, \hat{p}^m n \rangle \langle n, \psi_g^t \rangle}{\|\psi_g^t\|^2} \\ &= \lim_{t \rightarrow 0} \frac{\sum_n (nt)^m |\langle n, \psi_g^t \rangle|^2}{\|\psi_g^t\|^2} \\ &= p^m \end{aligned} \quad (\text{A.11})$$

where the first two lines are an expansion in terms of $|n\rangle$, the basis consisting of eigenvectors of \hat{p} - recall, that $U(1)$ momenta have discrete spectrum - , and the last line follows from our calculations above. This suggests that $|\langle n, \psi_g^t \rangle|^2 / \|\psi_g^t\|^2$ approaches - in the sense of distributions - just $\delta_{n,p/t}$, an observation that receives additional support from the explicit form of $\lim_{t \rightarrow 0} |\langle n, \psi_g^t \rangle|^2 / \|\psi_g^t\|^2$ that was calculated in [44]. That this also holds in a rigorous sense is guaranteed by the solution to the moment problem by Hamburger as quoted in the main text. In our case the a_n are given by the p^n , therefore $a_{n+m} = a_n a_m$ and thus the condition of the theorem is obviously satisfied. We can therefore conclude that our results for the integer powers of the momentum operator indeed determine $|\langle n, \psi_g^t \rangle|^2 / \|\psi_g^t\|^2$ to approach $\delta_{n,p/t}$. This important result will considerably simplify the calculations in the following subsections. We now come back to the problem of the roots of the momentum operator. Let m be an odd integer. Then

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{p}^{\frac{m}{2}} \psi_g^t \rangle}{\|\psi_g^t\|^2} &= \lim_{t \rightarrow 0} \frac{\sum_n \langle \psi_g^t, \hat{p}^{\frac{m}{2}} n \rangle \langle n, \psi_g^t \rangle}{\|\psi_g^t\|^2} \\ &= \lim_{t \rightarrow 0} \frac{\sum_n (nt)^{\frac{m}{2}} |\langle n, \psi_g^t \rangle|^2}{\|\psi_g^t\|^2} \\ &= p^{\frac{m}{2}} \end{aligned} \quad (\text{A.12})$$

where the last equality is now justified by the aforementioned theorem.

A.2 Expectation Values of the Holonomy Operator

In this subsection we will compute the semiclassical limit of expectation values of (powers of) the configuration operator \hat{h} . We can basically reduce this case to the one in the last subsection by the useful observation that $\hat{h} = e^{-1/2t} e^{-\hat{p}} \hat{g}$, see [44]. For m integer or half-integer we get

$$\begin{aligned}
\lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{h}^m \psi_g^t \rangle}{\|\psi_g^t\|^2} &= \lim_{t \rightarrow 0} \frac{e^{-m/2t} \langle \psi_g^t, e^{-\hat{p}} \hat{g} \dots e^{-\hat{p}} \hat{g} \psi_g^t \rangle}{\|\psi_g^t\|^2} \\
&= \lim_{t \rightarrow 0} \frac{e^{-m/2t} \langle \psi_g^t, e^{-m\hat{p}} \hat{g}^m \psi_g^t \rangle}{\|\psi_g^t\|^2} \\
&= \lim_{t \rightarrow 0} \frac{e^{-m/2t} \langle \psi_g^t, e^{-m\hat{p}} \psi_g^t \rangle g^m}{\|\psi_g^t\|^2} \\
&= \lim_{t \rightarrow 0} e^{-m/2t} e^{-mp} g^m \\
&= h^m
\end{aligned} \tag{A.13}$$

where we used in line two that all remaining commutator terms are at least of order t and therefore vanish in the limit $t \rightarrow 0$, and in line three that our coherent states are eigenstates of \hat{g} . It should be obvious from this that arbitrary mixed polynomials in \hat{p} and \hat{h} can be treated equivalently, to leading order in t .

A.3 Expectation Values of Commutators

In this subsection we intend to obtain the semiclassical limit of expectation values of commutator terms by direct computation. The main example we have in mind here is $\hat{h}^{-1}[\sqrt{\hat{V}}, \hat{h}]$ which plays an important role in the Hamiltonian constraint operator constructed in [21, 22]. Here \hat{V} denotes the volume operator. As this requires rather tedious calculations due to its structure, requiring at least a graph with three-valent vertices (in the gauge-variant case), we will restrict ourselves to the following case : we would like to check that $\hat{h}^{-1}[\sqrt{\hat{p}}, \hat{h}]/(it)$ has the right semiclassical limit, i.e. reproduces $h^{-1}\{\sqrt{|p|}, h\}$ which is $\text{isgn}(p)/(2\sqrt{|p|})$. We start with the observation that

$$\hat{h} \psi_g^t(h) = \sum_n e^{-n^2 t/2} g^n h^{-n+1} \tag{A.14}$$

We then have

$$\begin{aligned}
&\frac{\langle \psi_g^t, \hat{h}^{-1}[\sqrt{\hat{p}}, \hat{h}]/(it) \psi_g^t \rangle}{\|\psi_g^t\|^2} \\
&= -i \frac{\sum_n e^{-n^2 t} e^{2pn} (\sqrt{|(n-1)t|} - \sqrt{|nt|})/t}{\sum_n e^{-n^2 t} e^{2pn}} \\
&= -i \frac{\frac{1}{\sqrt{\pi}} \sum_n \int_{-\infty}^{\infty} dx \frac{(\sqrt{|x-s|} - \sqrt{|x|})}{s^{3/2}} e^{-x^2} e^{(2p/s - 2i\pi n/s)x}}{e^{\frac{p^2}{s^2}} \sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{s^2}}} \\
&= \frac{\frac{1}{\sqrt{\pi}} \sum_n e^{(2p/s - 2i\pi n/s)^2} \int_{-\infty}^{\infty} dx \frac{(\sqrt{|x+p/s - i\pi n/s - s|} - \sqrt{|x+p/s - i\pi n/s|})}{s^{3/2}} e^{-x^2}}{e^{\frac{p^2}{s^2}} \sum_n e^{-\frac{\pi^2 n^2 + 2i\pi np}{s^2}}}
\end{aligned} \tag{A.15}$$

where the last integral can involve the choice of a branch cut for $n \neq 0$. Since the integral certainly converges for any n and is multiplied by the exponentially fast vanishing function

$e^{-4\pi^2 n^2/s}$, by the argument already familiar from [41] for the limit $t \rightarrow 0$ it will be sufficient to keep the term $n = 0$ for what follows. We thus obtain for the expectation value

$$\lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{h}^{-1}[\sqrt{|\hat{p}|}, \hat{h}]/t \psi_g^t \rangle}{\|\psi_g^t\|^2} = -i \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{(\sqrt{|x+p/s-s|} - \sqrt{|x+p/s|})}{s^{3/2}} e^{-x^2} \quad (\text{A.16})$$

As we are ultimately only interested in the limit $t \rightarrow 0$, and therefore $s \rightarrow 0$, we aim at putting the integrand into a form that allows taking the $s \rightarrow 0$ limit inside:

$$\begin{aligned} & \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{(\sqrt{|x+p/s-s|} - \sqrt{|x+p/s|})}{s^{3/2}} e^{-x^2} \\ &= \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{(\sqrt{|xs+p-s^2|} - \sqrt{|xs+p|})}{s^2} e^{-x^2} \\ &= \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{(xs+p-s^2)^2 - (xs+p)^2}{s^2(|xs+p-s^2|^{1/2} + |xs+p|^{1/2})(|xs+p-s^2| + |xs+p|)} e^{-x^2} \\ &= \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{s^2 - 2(xs+p)}{(|xs+p-s^2|^{1/2} + |xs+p|^{1/2})(|xs+p-s^2| + |xs+p|)} e^{-x^2} \quad (\text{A.17}) \end{aligned}$$

It is easy to see that the limit $f_p(x)$ as $s \rightarrow 0$ of the integrand $f_p^s(x)$ exists pointwise. Furthermore it is clear that the modulus of the integrand is L^1 -integrable. Next, we write the integrand of (A.17) as $e^{-x^2} f_p^s(x) = e^{-x^2/2} g_p^s(x)$ and we seek to give a bound on $g_p^s(x)$ independent of s, x for s smaller than some s_0 . To that end we estimate $e^{-x^2/2} \leq 1$ for $|x| \leq 1$ and $e^{-x^2/2} \leq e^{-|x|/2}$ for $|x| \geq 1$ when estimating $g_p^s(x)$. Consider first the region $|x| \geq 1$. The first derivative of the estimated $|g_p^s(x)|$ then leads to a quadratic equation whose roots depend on the signs of both p, x . The local maxima turn out to lie at $x = \pm 1$ and $x \approx -p/s$. Only the former one is an absolute maximum. The value of $|g_p^s(x)|$ can then be estimated roughly by $1/\sqrt{|p|}$ up to a multiplicative, numerical constant and the same is true for the region $|x| \leq 1$. Altogether we have found, up to a numerical factor the following L_1 function, independent of s that dominates f_p^s

$$|f_p^s(x)| \lesssim \frac{e^{-x^2/2}}{\sqrt{|p|}} \quad (\text{A.18})$$

so that all conditions of the dominated convergence theorem are satisfied, and the $s \rightarrow 0$ limit can be taken inside the integral. We thus obtain

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{\langle \psi_g^t, \hat{h}^{-1}[\sqrt{|\hat{p}|}, \hat{h}]/t \psi_g^t \rangle}{\|\psi_g^t\|^2} \\ &= \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \lim_{s \rightarrow 0} \frac{s^2 - 2(xs+p)}{(|xs+p-s^2|^{1/2} + |xs+p|^{1/2})(|xs+p-s^2| + |xs+p|)} e^{-x^2} \\ &= \frac{-i}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \left(-\frac{1}{2\sqrt{|p|}} \text{sgn}(p) \right) e^{-x^2} \\ &= -\frac{1}{2\sqrt{|p|}} \text{sgn}(p). \quad (\text{A.19}) \end{aligned}$$

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